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PROPAGATION AND INTERACTION OF HYPERBOLIC PLANE WAVES IN NONLINEAR ELASTIC SOLIDS

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Introduction

1.1 Motivations

Waves are fascinating phenomena. We can encounter them almost everywhere. As Whitham writes in his classical book [168]:

> "... *flood waves in rivers, waves in glaciers, traffic flow, sonic booms, blast waves, ocean waves from storms, ... and waves in nonlinear optics or in various mechanical systems are of universal interest*".

According to Whitham, we can distinguish at least two classes of waves: *hyperbolic* and *dispersive*. They may eventually overlap, but each class has its own characteristic features. For *nonlinear hyperbolic* waves, formation of *shock waves* is a common phenomenon, while for *nonlinear dispersive waves*, propagation of *solitary waves* is typical.

Although it seems that there is no precise mathematical definition of a wave, one can roughly say, that this is a disturbance propagating with a *finite speed*. We will be mostly interested in mechanical waves. One can usually assign to each of the waves such attributes as e.g. a *phase*, *frequency*, *amplitude* etc.. Majority of waves belong to the class of dynamical phenomena, and a typical wave motion is governed by time dependent partial differential equations.

The importance of studying nonlinear mechanical waves is motivated by the fact that such waves are excellent tools for analyzing different properties of materials like e.g. features of damage (cracks, flaws, etc.). Waves can distort in nonlinear materials and create higher harmonics. Unlike as it is in the linear case, the *nonlinear waves* may interact resonantly producing new waves which propagate with phases of frequencies being a linear combination of the original ones. These phenomena are remarkably large in *damaged materials* in comparison to the undamaged ones (see e.g. [164] for details). Nonlinear wave diagnostics of damage is based on this fact. The *sensitivity of nonlinear methods* to the detection of damage features is much greater than that of linear acoustical methods. Nonlinear wave modulation spectroscopy (*NWMS*) is an example of a new nonlinear dynamic nondestructive evaluation technique (see e.g. again [164] for details).

In this work we attempt to create a rigorous mathematical apparatus which in future can be useful for such applications.

Mathematical models of wave type phenomena in real media, typically lead to very complicated nonlinear systems of partial differential equations. This is true in particular when there are several fields involved in the modelling process, like e.g. magneto-hydrodynamics, elasto-plasticity, magneto-elasticity etc. . In such cases the governing set of equations is usually too complicated to deduce from it analytically any qualitative results about its solution. Therefore in order to describe a nonlinear wave motion in such complicated cases, one may attempt to derive simpler models of a real phenomenon. Simpler but not oversimplified, so that we do not lose the essence of a nonlinear phenomenon like it may happen e.g. after linearization.

1.2 Aims and Methods

Our aim is to create a mathematical theory of nonlinear plane waves which could have applications in many branches of nonlinear material science. Using asymptotic expansions we derive simplified models to complicated, dynamical, nonlinear phenomena (including wave interactions), described originally by large systems of nonlinear partial differential equations. We are particularly interested in applications to elastic solids. The models which we derive are weakly nonlinear. Weak nonlinearity means that we are interested in small amplitude solutions. This is more refined than linearized models.

Asymptotic methods are very useful when we deal with complicated systems of *PDEs*. Weakly nonlinear geometric optics (*WNGO*) is suitable for nonlinear hyperbolic equations modelling wave type phenomena. The method works for *small amplitude* (weakly nonlinear) and *high frequency* waves. *WNGO* is based on the introduction of the additional "fast" independent variable and the use of a multiple scale analysis. It provides transport equations as the simplified asymptotic models for the evolution of wave profiles. The classical *WNGO* expansion results in the decoupled (in the nonresonant case) inviscid Burgers equations as the canonical asymptotic evolution equations for *strictly hyperbolic* and *genuinely nonlinear* waves. However in real models of continuum mechanics or physics it is quite common that the wave speeds may coincide and often they are not monotonic in certain directions so the assumptions of strict hyperbolicity and genuine nonlinearity are violated. This happens e.g. in nonlinear elastodynamics, the model which we are studying in the second part of this work.

In order to take into consideration the loss of strict hyperbolicity and loss of genuine nonlinearity we modify the classical *WNGO* expansion and derive new asymptotic evolution equations.

1.3 Main Results

The new results include:

- Derivation of a general structure of nonlinear plane waves' elastodynamics in terms of the strain energy function for an arbitrary direction of propagation, regardless of the anisotropy type.
- Presentation of explicit nonlinear plane waves' elastodynamics equations for a cubic crystal in the case of
  - an arbitrary direction of propagation and a geometrically nonlinear bur physically linear model,
  - an arbitrary direction of propagation in a cube face and both geometrically and physically nonlinear model,
  - three selected directions of pure mode propagation: $[1,0,0]$, $[1,1,0]$ and $[1,1,1]$ with the inclusion of higher order terms.
- Derivation of the simplified evolution equations for nonlinear elastic waves in the isotropic and the cubic crystal cases.
- Establishing a new model – the complex Burgers equation as the evolution equation describing interaction of a pair of transverse elastic waves propagating along the three-fold symmetry axis in a cubic crystal.[1]

[1] Our theoretical result (see Sec. 11.2 and also Domański [54], [56], [57]) was later experimentally confirmed in [66].

- Derivation of general analytical formulas for all interaction coefficients of nonlinear elastic plane waves propagating in any direction in an arbitrary hyperelastic medium.
- Calculations of all interaction coefficients analytically for all considered models.
- New formulations and interpretations of a null condition with the use of self–interaction coefficients are proposed.

We focused in particular on cases where a local loss of strict hyperbolicity and/or genuine nonlinearity occurs. We showed that this happens for transverse or quasi-transverse elastic waves and implies the presence of a cubic nonlinearity in the evolution equations for decoupled (quasi)–transverse waves. However, we found cases when (quasi)–transverse waves do interact at a quadratically nonlinear level for certain directions in anisotropic media. We showed that this happens e.g. for a cubic crystal for a $[1, 1, 1]$ direction. This fact is in contradiction to the isotropic case where it is known that shear elastic waves cannot interact at a quadratically nonlinear level see ([74]). We proved that the coupled systems with quadratic nonlinearity (complex Burgers equations) are new asymptotic models for interacting pairs of (quasi)-transverse waves propagating in the [1, 1, 1] direction in a cubic crystal. This is for the first time where complex Burgers equations appear in the context of nonlinear elasticity.

1.4 Acknowledgments

This work would not be possible without the influence of my home place – the Institute of Fundamental Technological Research of the Polish Academy of Sciences. The phenomenon of this Institute is that it embraces so many diverse people from different fields of science: engineers, mathematicians, physicists, both theoreticians and experimentalists as well as experts in scientific computing. Altogether they form a stimulating community and their scientific interaction produces a very creative atmosphere. I had a privilege and pleasure to spend many years in this place, listening to lectures, participating in seminars, talking and discussing with many people. It is impossible to mention all of them here.

The idea of applying asymptotic multi-scale methods in nonlinear elastodynamics was suggested to me by Professor Andrew Majda while I was visiting as a Post–Doctorate Fulbright Fellow the Department of

Mathematics and Program in Applied and Computational Mathematics at Princeton University in the Academic year 1989/90. The problem he proposed to me: to study stability and instability of multidimensional nonlinear wave patterns (in particular shock fronts) in nonlinear elasticity, turned out to be very difficult and subtle and it is still far from being solved satisfactorily. However, it encouraged me to undertake deeper studies in nonlinear elasticity and asymptotic methods. This work is a small outgrowth of my investigations.

Some part of this work was done while I was visiting Department of Mathematics of the Rostock University in Germany (March–September 1996). Special thanks are due to Professor Kurt Frischmuth for the invitation. The support from 'der Konferenz der Deutschen Akademien der Wissenschaften' is greatly acknowledged.

A substantial part was completed during my stay in the UK at the University of Cambridge. This was at the Department of Applied Mathematics and Theoretical Physics (DAMTP) and in Jesus College (July – August 1999) as well as in the Isaac Newton Institute of Mathematical Sciences (March – April 2003). The first visit was in the framework of Cambridge Colleges Hospitality Scheme and the second one was during the semester on Nonlinear Hyperbolic Waves in Phase Dynamics sponsored by AMAS and the Isaac Newton Institute. I am extremely and most grateful to the late Professor David G. Crighton the former Head of DAMTP and the former Master of Jesus College who in spite of his serious illness arranged my first visit to the University of Cambridge. Both DAMTP and Jesus College as well as the Isaac Newton Institute deserve my gratitude for providing me with excellent conditions to work and enjoy my stay in Cambridge.

I also thank Professor Shuichi Kawashima for inviting me to the Department of Mathematics at the University of Fukuoka in Japan (January – February 2001) and providing me with the latest computer equipment needed for some of my calculations.

Finally my one semester visit during the Academic Year 2001/2002 at the Department of Mathematics and Statistics of the University of Massachusetts at Amherst in the USA turned out to be very stimulating and fruitful. I thank my collaborator Professor Robin Young for the invitation and many discussions which clarified some aspects of this work.

Last but not least I thank Professor Ray W. Ogden for his warm encouragement and the invitation to the Department of Mathematics of the University of Glasgow in the UK (August 2003) where part of the work was done.

Part I

MATHEMATICAL THEORY

2

Hyperbolic Systems of Conservation Laws

In this chapter we review briefly a state of art of research in the selected parts of the mathematical theory of *nonlinear hyperbolic systems of conservation laws*. These systems create a framework which encompasses most of the dynamical models from continuum mechanics and physics, in particular the system of nonlinear elastodynamics discussed in Part II of this work. After some definitions concerning hyperbolic conservation laws, we discuss symmetric systems, well posedness and stability of shock waves. The emphasis is on less investigated degenerate systems. The results of the Lemma 2.8 in Sec 2.1.1 are new.

2.1 Hyperbolic Conservation Laws

This section contains a brief introduction to the mathematical theory of conservation laws. Systems of conservation laws arise as a result of macroscopic modeling of dynamical problems from continuum mechanics or physics. Conservation laws express the fact that the rate of change of the total amount of the density $\boldsymbol{u}$ of a macroscopic substance (e.g. mass, momentum or energy) contained in a fixed domain, say Ω, is equal to the flux of that substance $\boldsymbol{f}(\boldsymbol{u})$ across the boundary $\partial\Omega$ of this domain[1]:

$$\frac{\partial}{\partial t}\int_{\Omega} \boldsymbol{u}\, dV = -\int_{\partial\Omega} \boldsymbol{f}(\boldsymbol{u}) \cdot \boldsymbol{n}\, dS \tag{2.1}$$

with $\boldsymbol{n}$ being a unit outward normal to $\partial\Omega$.
The local *conservative form* of this equation for smooth $\boldsymbol{u}$ looks as follows:

[1] We disregard the source terms.

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0}, \tag{2.2}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ with t – a time variable and $\boldsymbol{x}$ – a vector space variable, $\nabla_{\boldsymbol{x}}\cdot$ denotes the divergence operator with respect to $\boldsymbol{x}$.

Examples of systems of conservation laws include Euler equations of gas dynamics, system of magnetohydrodynamics, equations of nonlinear elastodynamics, nonlinear optics and many more. The system of nonlinear elastodynamics will be discussed in details in part 2. In general (2.2) may be a system of m equation in n space variables and the number of dependent variables m not necessarily equals to the number of independent variables n. Using Cartesian coordinates we can write $\boldsymbol{x} = [x_1, .., x_n] \in \mathbb{R}^n$, $\boldsymbol{u} = [u_1, .., u_m] \in \mathbb{R}^m$, and $\boldsymbol{f}(\boldsymbol{u}) = [\boldsymbol{f}_1(\boldsymbol{u}), .., \boldsymbol{f}_n(\boldsymbol{u})] : \mathbb{R}^m \to (\mathbb{R}^m)^n$ with $\boldsymbol{f}_j(\boldsymbol{u}) : \mathbb{R}^m \to \mathbb{R}^m$ for $j = 1, 2, ..., n$.

The *quasilinear form* (assuming that the flux $\boldsymbol{f}(\boldsymbol{u})$ is differentiable, and using the chain rule) looks as follows

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x_j} = \boldsymbol{0}, \tag{2.3}$$

where the Jacobian $m \times m$ matrices

$$\boldsymbol{A}_j(\boldsymbol{u}) \equiv \nabla_{\boldsymbol{u}} \boldsymbol{f}_j(\boldsymbol{u}). \tag{2.4}$$

Consider plane wave solutions $\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{w}(t, \boldsymbol{x} \cdot \boldsymbol{k})$ of (2.3) for $\boldsymbol{k}$ with $|\boldsymbol{k}| = 1$. Let $\boldsymbol{x} \cdot \boldsymbol{k} = x$, then $\boldsymbol{w}(t, x)$ satisfies

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k}) \frac{\partial \boldsymbol{w}}{\partial x} = \boldsymbol{0}, \tag{2.5}$$

with

$$\boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k}) \equiv \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{w}) k_j. \tag{2.6}$$

In order that this plane wave solution is stable we require that the matrix $\boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k})$ has m real eigenvalues and is diagonalizable. Based on this, we introduce the following

Definition 2.1. *(Hyperbolicity). We say that the system (2.3) is* hyperbolic *if for any $\boldsymbol{w}$ and for all nonzero and real scalars k_j, the matrix (2.6) has m real eigenvalues and a complete set of eigenvectors.*

Definition 2.2. *(Strict Hyperbolicity). We say that the system (2.3) is* strictly hyperbolic *if it is hyperbolic and moreover all its eigenvalues are different for any* $\boldsymbol{w}$ *and for all nonzero and real scalars* k_j.

The pair which consists of the eigenvalue λ_j with the corresponding eigenvector $\boldsymbol{r}_j$ of the matrix (2.6) determines the j-th characteristic family of the system (2.5).

Definition 2.3. *(Genuine Nonlinearity). The j-th characteristic family is called genuinely nonlinear if, the j-th-eigenvalue* $\lambda_j(\boldsymbol{w}, \boldsymbol{k})$ *and the corresponding eigenvector* $\boldsymbol{r}_j(\boldsymbol{w}, \boldsymbol{k})$ *of the matrix (2.6), determined by*

$$\boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k})\boldsymbol{r}_j(\boldsymbol{w}, \boldsymbol{k}) = \lambda_j(\boldsymbol{w}, \boldsymbol{k})\boldsymbol{r}_j(\boldsymbol{w}, \boldsymbol{k})$$

satisfy

$$\nabla_{\boldsymbol{w}}\lambda_j(\boldsymbol{w}, \boldsymbol{k}) \cdot \Delta\boldsymbol{r}_j(\boldsymbol{w}, \boldsymbol{k}) \neq 0 \quad \textit{for any} \quad \boldsymbol{w} \quad \textit{and} \quad \boldsymbol{k} \quad \textit{with} \quad |\boldsymbol{k}| = 1. \tag{2.7}$$

Definition 2.4. *(Linear Degeneracy). The j-th characteristic family is called linearly degenerate if*

$$\nabla_{\boldsymbol{w}}\lambda_j(\boldsymbol{w}, \boldsymbol{k}) \cdot \boldsymbol{r}_j(\boldsymbol{w}, \boldsymbol{k}) = 0 \quad \textit{for any} \quad \boldsymbol{w} \quad \textit{and} \quad \boldsymbol{k} \quad \textit{with} \quad |\boldsymbol{k}| = 1. \tag{2.8}$$

Remark 2.5. G. Boillat [17] showed that if

$$\lambda_i(\boldsymbol{w}, \boldsymbol{k}) = \lambda_j(\boldsymbol{w}, \boldsymbol{k}) \quad \forall\boldsymbol{w} \quad \text{and} \quad \boldsymbol{k} \quad \text{with} \quad |\boldsymbol{k}| = 1 \quad \text{where} \quad i \neq j,$$

then

$$\nabla_{\boldsymbol{w}}\lambda_i(\boldsymbol{w}, \boldsymbol{k}) \cdot \boldsymbol{r}_i(\boldsymbol{w}, \boldsymbol{k}) = 0 = \nabla_{\boldsymbol{w}}\lambda_j(\boldsymbol{w}, \boldsymbol{k}) \cdot \boldsymbol{r}_j(\boldsymbol{w}, \boldsymbol{k}).$$

The general mathematical theory of conservation laws was developed mostly under the assumptions of *strict hyperbolicity* and *genuine nonlinearity.* These assumptions, however, are too restrictive to treat many of the applications. It turns out that in fact strictly hyperbolic systems are exceptions. Lax [110] and then Friedlands, Robbin, Sylvester [69] showed that for $m = 2, 3, 4, 5, 6$, in 3-D, any $m \times m$ hyperbolic system must be *nonstrictly hyperbolic.*[2] Therefore plane-wave solutions for such systems

[2] The fact that in 3-D, there are no $m \times m$ strictly hyperbolic systems occurs also e.g. for $m = 10, 11, 12$ etc. (see [110] and [69] for the details.)

in 3-D are described by one-dimensional hyperbolic systems with colliding eigenvalues. Our aim in this work is to study such systems, 'degenerate' from the point of view of mathematical properties and natural from the point of view of applications in continuum mechanics and physics, and in particular in elasticity. We begin with a 2×2 nonstrictly hyperbolic systems when strict hyperbolicity is lost at a single (umbilic) point.

2.1.1 Hyperbolic Umbilic Point

Let us consider the 2×2 system of conservation laws in one space dimension

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \boldsymbol{0}. \tag{2.9}$$

Definition 2.6. *(Umbilic point.) We call $\boldsymbol{u}_0$ an umbilic point if $\lambda_1(\boldsymbol{u}_0) = \lambda_2(\boldsymbol{u}_0)$, where $\lambda_1(\boldsymbol{u}), \lambda_2(\boldsymbol{u})$ are the eigenvalues of $\nabla_{\boldsymbol{u}}\, \boldsymbol{f}(\boldsymbol{u})$.*

Definition 2.7. *(Hyperbolic umbilic point.) Let $\boldsymbol{u}_0$ be an umbilic point. We call $\boldsymbol{u}_0$ a* hyperbolic umbilic point *if*

- $\nabla_{\boldsymbol{u}}\, \boldsymbol{f}(\boldsymbol{u})\big|_{\boldsymbol{u}=\boldsymbol{u_0}}$ *is diagonalizable, and*
- *there exists a neighbourhood Ω of $\boldsymbol{u_0}$ such that $\nabla_{\boldsymbol{u}}\, \boldsymbol{f}(\boldsymbol{u})$ has distinct eigenvalues for all $\boldsymbol{u}$ in $\Omega \setminus \{\boldsymbol{u_0}\}$.*

Lemma 2.8. *Suppose we have a hyperbolic system (2.9) in which*

$$\boldsymbol{f}(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} C(\boldsymbol{u}), \tag{2.10}$$

where

$$C(u_1, u_2) = \frac{1}{2}(\frac{1}{3}au_1^3 + bu_1^2u_2 + u_1u_2^2), \tag{2.11}$$

and a, b are real constants. Then $\boldsymbol{u}_0 = (u_1^0, u_2^0) = (0, 0)$ is a hyperbolic umbilic point.

Proof:
We have

$$\boldsymbol{f}(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} C(\boldsymbol{u}) = \frac{1}{2}(au_1^2 + 2bu_1u_2 + u_2^2, bu_1^2 + 2u_1u_2).$$

Hence

$$\nabla_{\boldsymbol{u}}\, \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} au_1 + bu_2 & bu_1 + u_2 \\ & \\ bu_1 + u_2 & u_1 \end{pmatrix}.$$

The eigenvalues of $\nabla_{\boldsymbol{u}} \boldsymbol{f}(\boldsymbol{u})$ are:

$$\lambda_{1;2}(u_1, u_2) \equiv \frac{1}{2}(A \mp \sqrt{B}) \tag{2.12}$$

where

$$\begin{aligned} A &\equiv (1+a)\,u_1 + b\,u_2, \\ B &\equiv \left((-1+a)^2 + 4\,b^2\right)\,{u_1}^2 + 2\,(3+a)\,b\,u_1\,u_2 + \left(4+b^2\right)\,{u_2}^2. \end{aligned} \tag{2.13}$$

The corresponding eigenvectors are

$$\boldsymbol{r}_{1;2}(u_1, u_2) = [\lambda_{1;2} - u_1, bu_1 + u_2] \tag{2.14}$$

We will show that $(u_1^0, u_2^0) = (0,0)$ is a *hyperbolic umbilic point*. First of all

$$\lambda_1(0,0) = \lambda_2(0,0) = 0.$$

So in fact $(0,0)$ is an umbilic point. Next we will show that $\lambda_1(u_1, u_2) \neq \lambda_2(u_1, u_2)$ for $(u_1, u_2) \neq (0,0)$. In order to prove this we will study the quadratic equation $B = 0$ with the unknown, say $u_2 \equiv X$, and parameters a, b, u_1. This equation reads

$$\left(4+b^2\right)\,X^2 + 2\,(3+a)\,b\,u_1\,X + \left((-1+a)^2 + 4\,b^2\right)\,{u_1}^2 = 0. \tag{2.15}$$

The discriminant to this quadratic equation is equal to

$$\Delta = -16(b^2 - a + 1)^2 {u_1}^2.$$

The solutions of the equation (2.15) are

$$X_{1,2} = -\frac{(a+3)bu_1 \pm 2\sqrt{-\left(b^2 - a + 1\right)^2 u_1^2}}{4+b^2}.$$

Hence provided $b^2 - a + 1 \neq 0$ and $u_1 \neq 0$, the quadratic equation (2.15) has got *no real roots*. Therefore, since $4 + b^2 > 0$ and $\Delta < 0$, so the left hand side of the quadratic equation (2.15) is positive and hence $B > 0$ for all a, b real such that $b^2 - a + 1 \neq 0$ and $u_1 \neq 0$ or $u_2 \neq 0$. This implies that $\lambda_1(u_1, u_2)$ and $\lambda_2(u_1, u_2)$ are real valued and

$$\lambda_1(u_1, u_2) \neq \lambda_2(u_1, u_2) \quad \text{for} \quad b^2 - a + 1 \neq 0 \quad \text{and} \quad u_1 \neq 0, u_2 \neq 0.$$

Therefore $\boldsymbol{u}_0 = (u_1^0, u_2^0) = (0,0)$ is the *hyperbolic umbilic point* for $a \neq b^2 + 1$.

Conversely we can also formulate the following lemma which follows from ([148]).

Lemma 2.9. *Let*

- $\boldsymbol{f} : \mathbf{R}^2 \to \mathbf{R}^2$ *be a quadratic function, and*
- $\boldsymbol{u}_0 = \mathbf{0}$ *– be an isolated hyperbolic umbilic point.*

Then there exist real a, b*, with* $a \neq 1 + b^2$ *such that* $\boldsymbol{f} = \nabla_{\boldsymbol{u}} C(\boldsymbol{u})$*, where*

$$C(u_1, u_2) = \frac{1}{2}(\frac{1}{3}au_1^3 + bu_1^2 u_2 + u_1 u_2^2).$$

From lemma 2.8 and lemma 2.9 it follows that (2.10) with (2.11) determines a canonical flux form for a 2×2 systems of conservation laws with a hyperbolic umbilic point.

Remark 2.10. The quasilinear systems of conservation laws with *hyperbolic umbilic points* play a very important role in applications. Schaeffer and M. Shearer [148] studied canonical forms for such equations with applications in *oil recovery* and in the appendix to [148] there is application to the *three-phase fluid flow in a porous media.* Later on (Sec. 10.3.2, see also W. Domański [53]) we show that such system play an important role in *elasticity* as well. In particular we prove that a complex Burgers equation (which has a flux with a special form of (2.11)) is a canonical model for a pair of quasi-shear elastic plane waves propagating along a $[1, 1, 1]$ three-fold symmetry axis in a cubic crystal. This result was generalized to arbitrary anisotropic media in a recent paper of W. Domański and A. Norris [65].

2.2 Symmetric Systems

The historical development of the theory of conservation laws was motivated mainly by applications in gas- and fluid-dynamics, (see e.g. the books of R. Courant and K. O. Friedrichs [34], P. D. Lax [108] or A. Majda [115]). Kurt Friedrichs was the first to observe that not only gas dynamics but most of the equations of continuum mechanics can be represented as *symmetric* or *symmetrizable* systems. The original Friedrich's definition concerns equations more general than the system (2.3).

Definition 2.11. *(Symmetric hyperbolicity). The quasilinear system of equations*

$$\boldsymbol{L}\boldsymbol{u} \equiv \boldsymbol{A}_0(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial t} + \sum_{j=1}^{n} \boldsymbol{A}_j(\boldsymbol{u}) \frac{\partial \boldsymbol{u}}{\partial x_j} = \mathbf{0} \tag{2.16}$$

is symmetric hyperbolic *if*

- *all matrices* $\boldsymbol{A}_j(\boldsymbol{u})$ *are symmetric for* $j = 0, 1, ...n$,
- *the matrix* $\boldsymbol{A}_0(\boldsymbol{u})$ *is positive definite.*

Remark 2.12. For the system (2.3) (where $\boldsymbol{A}_0(\boldsymbol{u}) = \boldsymbol{I}$ - the identity matrix) this definition obviously reduces to only one requirement, namely the symmetry of the matrices $\boldsymbol{A}_j(\boldsymbol{u})$ with $j = 1, ...m$.

Remark 2.13. It happens quite often in applications that in spite of the fact that the original system (2.3) is *not* symmetric, there exists a positive definite matrix, say S, such that when multiplied by S, the system becomes symmetric hyperbolic. In such a case the system (2.3) is called *symmetrizable.*

2.2.1 Symmetrization

The idea of symmetrization of hyperbolic partial differential equations comes from S.K. Godunov [73] who was the first to symmetrize the equations of gasdynamics. Later Friedrichs and Lax [70] generalized Godunov's idea to abstract hyperbolic systems of conservation laws. The Hessian of a *mathematical entropy function* serves as a symmetrizer.

Definition 2.14. *(Mathematical entropy function). A scalar-valued function* $\eta(\boldsymbol{u}) : \mathbb{R}^m \to \mathbb{R}$ *is called a mathematical entropy function associated with (2.2) or (2.3), if there exists a vector-valued function (entropy flux)* $\boldsymbol{q}(\boldsymbol{u}) = [\boldsymbol{q}_1(\boldsymbol{u}), .., \boldsymbol{q}_n(\boldsymbol{u})]$ *with* $\boldsymbol{q}_j(\boldsymbol{u}) : \mathbb{R}^m \to \mathbb{R}^m$ *for* $j = 1, 2, ..., n$*, such that*

$$(\nabla_{\boldsymbol{u}}\eta)(\nabla_{\boldsymbol{u}}\boldsymbol{f}_j) = \nabla_{\boldsymbol{u}}\boldsymbol{q}_j.$$

Definition 2.15. *(Strictly convex entropy). The entropy function* $\eta(\boldsymbol{u})$ *is called strictly convex if its Hessian is strictly positive definite, that is if*

$$\boldsymbol{u} \cdot \boldsymbol{H}\boldsymbol{u} > 0 \quad \textit{for any} \quad \boldsymbol{u} \neq \boldsymbol{0},$$

where $\boldsymbol{H} \equiv \nabla_{\boldsymbol{u}}(\nabla_{\boldsymbol{u}}\eta(\boldsymbol{u}))$.

Theorem 2.16. *(Symmetrization). Suppose* $\eta(\boldsymbol{u})$ *is a strictly convex entropy function associated with the systems (2.2) or (2.3), then*

$$\boldsymbol{H}\boldsymbol{A}_j = \boldsymbol{A}_j^T\boldsymbol{H},$$

so the systems (2.2) or (2.3) are symmetrizable.

Definition 2.17. *(Entropy solution). A function* $\boldsymbol{u}$ *is an entropy solution of (2.2) if it satisfies (2.2) and for all convex entropy functions,* $\eta(\boldsymbol{u})$*, an additional inequality is satisfied,*

$$\frac{\partial \eta}{\partial t}(\boldsymbol{u}) + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{q}(\boldsymbol{u}) \leq 0.$$

The basic questions regarding the existence, uniqueness and stability of entropy solutions for general nonlinear hyperbolic systems are open. The notion of the entropy solution is intimately related to the notion of a viscosity limit solution which was introduced to single out a *unique* 'physically relevant' weak solution of (2.2) by the limiting procedure.

Definition 2.18. *(Viscosity solution). A function* $\boldsymbol{u}$ *is a viscosity solution of (2.2) if* $\boldsymbol{u} = lim\, \boldsymbol{u}^{\varepsilon}$ *for* $\varepsilon \to 0$*, where* $\boldsymbol{u}^{\varepsilon}$ *satisfies*

$$\frac{\partial \boldsymbol{u}^{\varepsilon}}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{u}^{\varepsilon}) = \varepsilon\, \nabla_{\boldsymbol{x}} \cdot (Q\, \nabla_{\boldsymbol{x}} \boldsymbol{u}^{\varepsilon}), \quad \textit{with} \quad Q\, \varepsilon > 0. \tag{2.17}$$

The general advantage of the symmetrization is that we automatically obtain the local well-posedness of the Cauchy problem for the symmetrized system.

2.3 Well-posedness

Well-posedness of a given problem means existence, uniqueness and stability of the solution to this problem. The information about well-posedness is crucial for the model. Before presenting theorems about the initial-value problems for nonlinear hyperbolic systems, we will introduce some functional spaces and norms. It is important which space will be selected and in what kind of topology the chosen space is equipped, since this determines the notion of convergence.

2.3.1 Functional Spaces and Norms

L^p and Sobolev Spaces

The space of square integrable functions L^2 is very useful in linear theory. It has a physical interpretation of finite energy. Estimates of solution in L^2 norm give well-posedness of a weak solution. Thanks to Friedrichs mollification technique, in order to obtain well-posedness of the classical solution one needs only estimates in Sobolev spaces. In other words one has to estimate the derivatives of the solution in the L^2 norm.

Let us now define the Sobolev spaces H^m. The function $f \in L^2$ belongs to the Sobolev space H^m if it is "m"-times weakly differentiable, and if each of the weak derivative $f^{(j)} \in L_2,\ j = 1, \ldots, m$. In particular $H^0 = L^2$. H^m is naturally equipped with the associated norm

$$||f||_{H^m} = \|f\|_{L^2} + \|f^{(m)}\|_{L^2}.$$

We define also the L^p norms:

$$||f||_{L^p} = \begin{cases} (\int |f(x)|^p \, dx)^{1/p}, & if\ 1 \le p < \infty, \\ \operatorname{ess\,sup}_x |f(x)|, & \text{if } p = \infty. \end{cases}$$

Analogously to H^m, we can define the Sobolev space $W^{m,p}$. The function $f \in L^p$ belongs to the Sobolev space $W^{m,p}$ if it is "m"-times weakly differentiable, and if each of the weak derivative $f^{(j)} \in L^p,\ j = 1, \ldots, m$. We have in particular that $W^{m,0} = H^m$ and $W^{0,p} = L^p$. $W^{m,p}$ is naturally equipped with the associated norm

$$||f||_{W^{m,p}} = \|f\|_{L^p} + \|f^{(m)}\|_{L^p}.$$

Linear Theory

The problem of well-posedness of the initial-value problem (IVP) or initial boundary-value problem (IBVP) for *linear* hyperbolic P.D.E.'s is well understood thanks to the deep results from functional analysis, like e.g. Riesz representation theorem and Hahn-Banach theorem (see e.g. [40]). Energy (a priori) estimates in L^2 for the given and the adjoint problems with the use of the above mentioned theorems allow to get an existence of a *weak* solution in L^2 space for a given problem. Then similar estimates in Sobolev spaces together with the Friedrich's mollifying technique imply the regularity and a *global* in time existence of a *classical* solution to a linear hyperbolic problem [40].

Unfortunately things are much more complicated in the nonlinear case, where a weak convergence does not easily imply a strong convergence.

The space of functions of bounded variation

For nonlinear equations in 1-D, the space of functions of a bounded variation is very useful. Functions of the bounded variation need not be that smooth, they may have kinks and jumps.

Definition 2.19. *A function $f(x)$ is said to have a bounded variation if, over the closed interval [a,b], there exists an M such that $|f(x_1) - f(a)| + |f(x_1) - f(x_2)| + \cdot|f(x_{n-1}) - f(b)| \leq M$ for all $a < x_1 < ...x_{n-1} < b$.*

The space of functions of a bounded variation is denoted "BV," and can be equipped with a seminorm. The seminorm is equal to the supremum over all sums above.

2.3.2 Local Well-posedness for Symmetric Hyperbolic Systems

The symmetry assumption allows to use energy estimates to show existence and uniqueness of a local in time, classical solution to the Cauchy problem for (2.16). The idea of the proof of a local existence for quasilinear equations in general Banach spaces, with the help of a contraction principle and a fixed point theorem, using the iteration scheme, goes back to the Polish mathematician J. P. Schauder [146]. Its more modern versions with applications to quasilinear hyperbolic first order systems can be found in P. D. Lax [108].[3]

Theorem 2.20. *(Local well-posedness). Consider the Cauchy problem*

$$\begin{cases} \boldsymbol{Lu} = \boldsymbol{0} \\ \boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \end{cases} \tag{2.18}$$

for a quasilinear symmetric hyperbolic system (2.16). Assume that $\boldsymbol{u}_0 \in H^s(\boldsymbol{R}^n)$ for $s > \frac{n}{2} + 1$. Then there is a time interval $[0, T]$ with $T > 0$, so that the Cauchy problem (2.18) has a unique classical solution $\boldsymbol{u}(t, \boldsymbol{x}) \in C^1([0, T] \times \boldsymbol{R}^n)$.

Remark 2.21. A similar theorem holds for nonhomogeneous system $\boldsymbol{Lu} = \mathbf{f}$ (with a nonzero right hand side).

The theorem gives only *local* existence of a unique smooth solution for a quasilinear symmetric hyperbolic system. Typically for hyperbolic nonlinear equations there is a time $T_\star$ so that $\boldsymbol{u}$ or its derivatives blow up as $t \to T_\star$.

[3] see also T. Kato [97], A. Majda[115], M. Taylor [160] or R. Racke [136].

2.3.3 Blow-up

From the above theorem it follows that the *classical solution* of the Cauchy problem for nonlinear symmetric hyperbolic systems exists usually only *locally in time*. Even when we start from smooth initial data, singularities typically develop in a finite time. This breaking of the classical solution can be connected with the formation of shock waves and forces to enlarge the class of admissible solutions for nonlinear problems to include weak solutions containing discontinuities. Unfortunately, weak solutions are in general not unique, hence we have to impose additional admissibility criteria to single out physical solutions. Such criteria for physical systems are typically in the form of some kind of entropy inequalities. We will briefly discuss different admissibility criteria in the section devoted to shock waves.

The situation is even worse if we are dealing with at least 3×3 systems and periodic or quasi (almost) periodic data. What plays a crucial role in such cases is a resonant interaction of waves which may have a very destabilizing effect. During such interactions wave amplitudes may magnify and a finite time blow up of either the BV or L^∞ norms of the solution may take place. The examples of blow up can be found e.g. in [87], [95], [171] and [172].

A crucial role in analyzing the criteria for blow up are played by the interaction coefficients. In part two we will calculate explicitly in the analytical form all these interaction coefficients for the system of elastodynamics in both an isotropic and a cubic crystal cases. Using the self–interaction coefficients we will also formulate criteria for global existence of a classical solution in the isotropic case.

2.3.4 Nonlinear Theory

One of the methods of proving well–posedness for nonlinear hyperbolic equations is to consider their viscous or dispersive approximations to avoid problems with discontinuities. Such effects like dispersion or viscosity smooth out the discontinuous solutions of nonlinear hyperbolic problems. Next, one tries to use the limiting procedure, taking the dispersion or viscosity to zero, however, rapid oscillations and concentrations of solutions encountered in this process may cause serious problems.

Moreover in nonlinear problems in multi-dimensions we still do not know which functional spaces are the most appropriate for studying well-posedness. So far general results concerning the existence of weak solutions for systems of conservation laws are restricted to one space dimension. They rely on two basic assumptions concerning the structure of the eigenvalues of the hyperbolic system: *strict hyperbolicity*, that is distinctiveness of the eigenvalues, and *genuine nonlinearity*, that is monotonicity of the eigenvalues in the direction of the corresponding right eigenvector. Under these assumptions global in time existence of weak solutions to the Cauchy problem with small and of bounded variation (BV) initial data was proved (J. Glimm [71]). Similar results were also obtained by S. Bianchini and A. Bressan [13], (see also A. Bressan [21]) with the help of a vanishing viscosity approximation. Their achievement was to obtain for the first time the total variation estimates directly from the vanishing viscosity method.

On the other hand there is a functional analytical approach based on weak convergence. This approach was used by L. Tartar, F. Murat, R. DiPerna. The main difficulty in proving the well-posedness of nonlinear partial differential equations is the fact that nonlinear functions or functionals need not to be continuous in the weak topology. This manifests itself e.g. in difficulties in converting weak convergence into strong convergence and passing to the limit in nonlinear problems. The functional analytic methods require in such cases uniform bounds on *all* partial derivatives. This is extremely difficult to obtain.

The development of the method of *compensated compactness* by French mathematicians L. Tartar and F. Murat in the late seventies brought some hope (see e.g. [159]. The advantage of this method lies in the fact that it allows to pass to the limit in the nonlinear problems using only the derivative control presented by the special linear combinations of nonlinear functions. However so far only some scalar and 2×2 systems have been treated with the help of this method, see however [157]. Measure valued solutions [**?**],[121], Young measures [125], kinetic formulation and divergence–measure fields [30] are some of the modern and promising tools in studying nonlinear hyperbolic conservation laws.

2.4 Shock Waves

Shock waves are special discontinuous solutions of hyperbolic conservation laws. The development of the theory of shock waves as well as conservation laws began from the investigations in compressible gas and fluid dynamics. The classical book of Richard Courant and Kurt Friedrichs "Supersonic Flow and Shock Waves" [34] deals mostly with compressible gases in one space dimension. One of the central problems in the theory of shock waves is to define the class of admissible (stable) shocks. Currently the theory is not yet completely developed and again as it was emphasized in the previous section, most open problems remain and difficulties arise with degenerate systems.

In the mathematical literature (see e.g. A Majda [115]), shock waves are defined as the special piecewise smooth discontinuous solutions of the first order hyperbolic systems of conservation laws. According to this definition, a shock wave is determined by a triple: a smooth *singular surface* $\mathcal{S}$ across which the conserved quantity $\boldsymbol{u}$ suffers a jump, and *two functions* $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ defined in respective domains Ω^+ and Ω^- on either side of this surface. The functions $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ are smooth solutions of the quasilinear form of conservation laws in Ω^+ and Ω^- respectively. Moreover, they satisfy the *Rankine-Hugoniot* conditions which relate the values of the jumps in the conserved quantity and in the corresponding fluxes, to the shock speeds. Besides, shocks should satisfy certain stability requirements which usually follow from the second law of thermodynamics and are connected with the growth of the entropy.

2.4.1 Lax Condition

As we have mentioned the first mathematical papers devoted to shock stability were concerned with gas and fluid dynamics. Based on the applications in these fields, P. D. Lax [108] formulated a general stability condition for plane waves in hyperbolic $m \times m$ systems. Lax condition generalizes the *supersonic* ahead and *subsonic* behind the shock front, stability requirements known for gases. This condition specifies inequalities which every shock speed σ must satisfy to define a stable (compressive) shock front. Namely, the shock speed σ should be such that $\lambda_k^+ < \sigma < \lambda_k^-$, and $\lambda_{k-1}^- < \sigma < \lambda_{k+1}^+$, where $\lambda_k^\pm$ are linearized wave speeds of a k-wave (evaluated at the appropriate side of the discontinuity). In other words, Lax condition determines the precise number of incoming and outgoing characteristics from the left and right hand side of the shock front.

For systems which come from continuum mechanics these conditions have a natural interpretation connected with the entropy function, namely that the material which crosses the discontinuity should suffer the increase of an entropy. For classical compressible fluids e.g. a perfect gas Lax condition and the entropy condition are equivalent.

2.4.2 Nonclassical Shocks

However, it turned out that even in fluid dynamics there are shock waves that do not satisfy Lax conditions. This is typical for nonclassical fluids like e.g. van der Waals fluids, superfluid helium, and some dense gases called Bethe-Zeldovich-Thompson (BZT) gases. H. Bethe [12](1942) and Y. B. Zeldovich [177](1946) were the first to suggest the existence of dense gases and it was P. A. Thompson [162](1971) who was the first to emphasize the role of a certain thermodynamic parameter called a fundamental derivative Γ in characterizing dense gases. For a perfect gas Γ is always positive. However there are some dense gases for which at some pressures and temperatures Γ may become negative. In these region of negative nonlinearity, wave disturbances steepen backwards to form expansion shocks. This is in contrast to a classical case where disturbances steepen forward to form compression shocks. Such *non-classical shocks* like the one in dense gases violate Lax condition. They are characterized as shocks with too many incoming into the shock front characteristics (so called *overcompressive shocks*) or too few incoming characteristics (*undercompressive shocks*). Non-classical shocks may also appear in solids e.g in such cases where the stress-strain relation is nonconvex .

This loss of convexity in the flux function, brings mathematical difficulties in the analysis of shocks. Analytically this is connected with (local) loss of *strict hyperbolicity*, or (local) loss of *genuine nonlinearity* or when the equations change type. This last situation physically may be related e.g. to the phenomenon of phase transition.
In all of the above mentioned situations it may turn out that Lax conditions are not applicable. There is a number of different admissibility conditions in the mathematical literature e.g. evolutionary condition (Jeffrey and Taniuti [86]), Liu condition and many others. These conditions work for some particular cases but no general mathematical theory has been developed so far to handle the problem of stability of all nonclassical shocks.

What are then the correct stability criteria for nonclassical shocks? It turns out that it is not possible to solve this problem on a purely hyperbolic ground. Since a mathematical idealization is not enough, one has to consider a more physically realistic situation and allow dissipation or dispersion or both to be present in the equations. Then using e.g. the *viscosity* or the *viscosity - capillarity criterion* one can derive some stability results. This criterion is based on studying travelling waves solutions to the conservation laws enhanced by the viscous and strain gradient terms. The problem of the stability of shocks is reduced to the existence of trajectories (viscous profiles) joining the critical points of the obtained dynamical system. Undercompressive shocks are very difficult to investigate because they correspond to saddle - saddle connections which are very subtle to study. To deal with such shocks an additional admissibility condition associated to the diffusive-dispersive model was introduced. This condition is called the *kinetic relation* (see e.g. [1] or [111]).

2.4.3 Multidimensional Shocks

S.P. Dyakov in 1954 [66] and then V. M. Kantorovich in 1957 [101] formulated the conditions for instability of shock waves in gases in multidimensions as well as for the spontaneous emission of sound and entropy waves. The first rigorous mathematical theory of stability of multidimensional shock waves for general hyperbolic systems of conservation laws were established in the eighties by A. Majda [115, 116, 117] This theory is based on the transformation of a free boundary value problem to a fixed domain, linearization and analysis of the obtained IBVP for a linear hyperbolic system. The stability of shock is equivalent to the well–posedness of the linearized IBVP, which holds provided one can prove $L^2(H^s)$ estimate of the solution up to the boundary in terms of the $L^2(H^s)$ norms of the data. The energy inequalities were derived using Kreiss symmetrizer technique [104] with the help of of pseudodifferential operators under the *uniform Lopatinskii (Kreiss)* condition. This condition can be also characterized purely algebraically as positive definiteness for a range of some parameters of a certain determinant called the *Lopatinskii determinant.*

This mathematical theory, which was motivated by the applications in fluid dynamics, although pretty general does not handle many other applications. The reason is the loss of derivatives in the energy estimates which occurs due to loss of strict hyperbolicity and convexity but also because the uniform Lopatinskii condition is too restrictive to handle typical physical boundary conditions (see Domański [41]).

First, multiplicities of eigenvalues in the original system may cause problems, because they may lead to the loss of the derivatives at the boundary. This in turn implies difficulties in generalizing the estimates from a linear to nonlinear problem. However (see Métivier [123, 124]) for so called symmetrizable systems everything is fine. We may construct the symmetrizer and derive the energy estimates even for a uniformly characteristic boundary.

Another even more important issue is the fact that Kreiss well posed problems do not handle most of the typical physical boundary conditions. Directions along which Kreiss condition fails are called resonant. Majda and Artola [5], using weakly nonlinear geometrical optics, investigated stability and instability of shock waves for resonant directions in fluids. These are the only results known to the author on applications of *WNGO* to multidimensional nonlinear waves stability which still remains an open problem.

The failure of the uniform Lopatinskii (Kreiss) condition is connected with the appearance of interesting phenomena like e.g. the so called spontaneous emission, formation of rarefaction shock waves and shock splitting. A van der Waals fluid is one of the examples of an acoustic emission material having such anomalous behavior. Acoustic emission instability can be treated also as a resonant reflection when for a certain angle of incidence the ratio of reflected to incident waves becomes infinite. There is no mathematical theory describing rigorously these phenomena in multidimensions especially for solids.

To treat a shock wave as a discontinuous solution of a hyperbolic system of conservation laws is a mathematical idealization. A more realistic approach is to include e.g. viscous and heat conduction effects and to study dissipative shocks.

3

Asymptotic Expansions

3.1 Introduction

Perturbation or asymptotic methods are powerful tools in many problems of applied mathematics and engineering. The methods are based on the identification in (or introduction into) the problem of a small parameter, typically denoted by ϵ, with $0 < \epsilon << 1$ (occasionally we may need two or more small parameters). Usually there is an easily obtained solution of a problem when $\epsilon = 0$, e.g. a trivial null solution of a homogeneous equation. One then uses this solution to construct a linearized theory about it. The resulting set of equations is then solved consecutively, giving a solution which is valid locally near $\epsilon = 0$. As Holmes writes [80]

> "*The principle objective when using perturbation methods is to provide a reasonably accurate expression for the unknown solution. By doing this, one is able to derive understanding of the physics of the problem*".

3.1.1 Geometric Optics

Among the asymptotic methods applied in the wave theory, one of the most effective is that of *geometric optics*. The name geometric optics was first associated only with the propagation of light. The visible light propagates along local paths called rays with a very short wavelength. Therefore geometric optics is applicable to problems concerning waves propagating with short wavelength and high frequency. This method has found applications in different fields and is known e.g. in quantum mechanics under the name of *WKB*[1].

[1] The letters WKB stand for the abbreviation of the names Wentzel–Kramers–Brillouin.

In mathematical literature the term geometric optics is used as a synonym of the short wavelength asymptotic analysis of solutions of general (mainly hyperbolic) systems of partial differential equations (*PDE*s). Using this method in a *linear case* one can reduce systems of complicated *PDE*s to much simpler *eikonal equations* for the unknown phases and to the ordinary differential equations (*ODE*s) – *transport equations* along the rays for the unknown amplitudes. Unfortunately a direct application of this method to the nonlinear equations fails. In order to extract nonlinearity and place it into the evolution equations, we need to use a multiple scale method.

3.1.2 Multiple Scales Methods

Different time and space scales are common in nature. To say that a process lasts long requires first a definition of a time scale and to say that something is large, requires a definition of a length scale.

As we have mentioned, the asymptotic method of multiple scales is used in a nonlinear theory to extract 'essential' nonlinearity from the original highly nonlinear system. It can be very useful e.g. in material science.

As Glimm and Sharp [72] write:

> "... *the prediction of macroscopic properties of solids on the basis of their microstructural state combined with their atomic level composition is a grand dream for material science, culminating with the program of materials by design. Such macroscopic properties of solids as e.g. strength, toughness or electrical conductivity depend crucially on structures at various length scales. For example, the strength of a pure crystal is typically an order of magnitude larger than the strength of a polycrystal formed from the same material, while the fracture toughness and ductility are much lower. Thus metals are inherently multiscale. Plastic deformation of metals has its origin in the flow of dislocations. But dislocation theory by itself will not predict plastic flow rates and yield strength* ..."

Therefore we need to study the interaction or influence of phenomena on one scale on those of another. Asymptotic methods of multiscale expansion might be very effective tools in realizing this goal. A very important application of the method of multiple scales is for heterogeneous materials such as composites, laminated structures, porous media etc. . An approach to these problems is a homogenization. Homogenization is a two scale theory in which typically the microscale structure is assumed to be periodic.

The method successfully predicts the form of averaged equations, but is less successful in determining the numerical values of parameters which appear in these equations.

In the next section we present the method of weakly nonlinear geometric optics which combines multiple scales with geometric optics and is used for nonlinear wave type phenomena.

3.1.3 Weakly Nonlinear Geometric Optics

Weakly Nonlinear Geometric Optics (*WNGO* in short) is particularly useful when we study a nonlinear hyperbolic wave process. *WNGO* differs from the linear geometric optics in that the transport equations for the amplitudes of waves in *WNGO* theory are nonlinear *PDE*s. They are nonlinear but much simpler than the original ones. Moreover in the simplest case (e.g. for initial data of compact support in phase variables) we get the decoupled transport equations and each equation for the amplitude of each wave, so instead of studying a whole system, we may restrict ourselves to study single canonical equations. As typical canonical equations we get e.g. inviscid Burgers equations for genuinely nonlinear hyperbolic problems, Burgers equations for systems with dissipations or KdV equations for certain problems in which dispersion is present. Of course these canonical evolution equations are better understood than the original ones. Moreover, the exact solutions for the above canonical equations can quite often be found analytically. The analysis of the properties of these canonical equations gives us qualitative information about the original problems.

In *WNGO* we assume that the waves have small amplitudes and wavelenghts much smaller than the size of a region over which the problem is considered. Since the wavelength of visible light is of order of magnitude 0.005 cm, geometrical optics is good to describe optical phenomena and actually first applications of this theory were initially applied in the field of optics, hence the name. The method however has a more universal character and may be applied in many different problems, which will be shown in this work.

The brief description of the idea is the following. Suppose that we have a certain constant state $\mathbf{u}_0$ and assume that we perturb it by a small amplitude and high frequency disturbance. This disturbance is then propagated by conservation laws which describe the dynamics of the process. We then expand the flux matrices of the conservation laws in terms of the small parameter ϵ. Next we collect the terms of alike powers of ϵ. As a result we first obtain the eikonal equations for the unknown phases and next the transport equations for the unknown amplitudes.

More precisely the aim is to simplify the original nonlinear hyperbolic system on the expense of its generality, that is to reduce a complicated model to a simpler one by slightly restricting its range of applicability (assumptions about a small amplitude and high frequency expansion).

The idea is to consider waves with wavelength d much smaller than the size L of the region over which the wave is travelling (a typical propagation distance). So we are dealing with "short waves" which propagate along rays.

Let $\epsilon = \frac{d}{L}$ be a small parameter in the expansion series. Typically before applying the asymptotic expansions, we need to nondimensionalize all the physical quantities appearing in the model, simply to know the order of magnitudes of some of these quantities. Since hyperbolic equations are scale invariant, so there is no small parameter 'naturally' present in the equation. Therefore we introduce it into the initial data.

We perturb a constant state solution $\mathbf{u_0}$ in the following way:

$$\mathbf{u}_0^\epsilon(0,x) = \mathbf{u}_0 + \epsilon \mathbf{u}_1(0,x,\frac{x}{\epsilon}) \tag{3.1}$$

here $\frac{x}{\epsilon} = \theta$ is a high frequency ('fast') variable.

The introduction of the fast variable together with the method of multiple scale will assure the presence of nonlinearity in the evolution equations. In the nonresonant case we assume that the function $\mathbf{u}_1(0,x,\theta)$ has compact support in θ. This assumption prevents the occurrence of a resonance. For the resonant interactions we assume that the function $\mathbf{u}_1(0,x,\theta)$ has period T in θ for each x. Then $\mathbf{u}_0^\epsilon(0,x)$ has an amplitude of order ϵ, and oscillates with order ϵT.

History and Advantages of WNGO

The origins of the method of *WNGO* come from the heuristic works of physicists who, like e.g. Lev Landau applied this method already in the forties [106] (see also M. J. Lightill [112]). Among the physicists the method was known under the name of "reductive perturbation method" and was used intensively in different physical contexts, see e.g. the papers of T. Taniuti [158]. However the physicists did not care about proving the validity of this method, neither did they attempt to check how close the "asymptotic solution" was to the real solution.

The more rigorous mathematical approach began with the work of Y. Choquet–Bruhat from 1969 ([31]), but the true development of *WNGO* started in the eighties after the PhD thesis of John Hunter written under the supervision of Joe Keller [84]. Then a number of papers appeared in a short time [118], [85], [119]. The three names of people who were most active in this area of research at that time should be mentioned. These are: John Hunter, Andrew Majda and Rodolfo Rosales. These three applied mathematicians working mainly together established a foundation of modern theory of *WNGO*. This theory is still a very active research area. Many new papers appear. However, since the motivation of application of *WNGO* was fluid dynamics, so the developed theory was mainly applied to strictly hyperbolic systems. This is not the case in solids, and in particular in elastodynamics and also in magneto-elastodynamics. In this work we modify the *WNGO* expansion so that it is applicable in these cases.

WNGO is a rigorous mathematical theory which is applicable not only to harmonic, but to arbitrary waves. Using this method one can treat multiwave interactions and resonances as well. It handles waves in multidimensions. The method of *WNGO* can be applied to inhomogeneous media, and it allows to include such effects like dissipation and dispersion.

3.2 Classical Asymptotics

In this section we begin the presentation of the method of weakly nonlinear geometric optics (*WNGO*). First, we derive the asymptotic evolution equations for small amplitude noninteracting weakly nonlinear waves. The form of the evolution equations depends heavily on the eigenstructure of the linearized systems. While standard–classical geometric optics expansion suffices to derive nonlinear evolution equations for the genuinely nonlinear (e.g. longitudinal elastic) waves, modifications of the classical expansion are needed for linearly degenerate (e.g. transverse elastic) waves. These modifications will be presented in Sects. 3.3 - 3.6.

Here, for completeness, we briefly recall the classical *WNGO* expansion. First we consider the simplest case of a single wave expansion. Let us consider an initial-value problem (*IVP*) for a quasilinear hyperbolic system:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}^\epsilon}{\partial t} + \boldsymbol{A}(\boldsymbol{u}^\epsilon)\,\dfrac{\partial \boldsymbol{u}^\epsilon}{\partial x} = \boldsymbol{0} \\ \boldsymbol{u}^\epsilon(0,x) = \boldsymbol{u}_0 + \epsilon\,\boldsymbol{u}_1(0,x,x/\epsilon) \end{cases} \tag{3.2}$$

ϵ – a small parameter, $\boldsymbol{u}_0$ – a constant state solution. The initial data are supposed to have a compact support. This assumption prevents resonant interactions from occurring.

A weakly nonlinear geometric optics asymptotic solution to the *IVP* (3.2) around $\boldsymbol{u}_0$ is sought in the form:

$$\boldsymbol{u}^\epsilon(t,x) = \boldsymbol{u}_0 + \epsilon \boldsymbol{u}_1(t,x,\eta) + \epsilon^2 \boldsymbol{u}_2(t,x,\eta) + \mathcal{O}(\epsilon^3) = \boldsymbol{u}_0 + \epsilon \tilde{\boldsymbol{u}}(t,x,\eta) \quad (3.3)$$

with $\eta = \epsilon^{-1}(x - \lambda_j t)$.

We assume:

1) Taylor expansion of the flux[2]:

$$\boldsymbol{A}(\boldsymbol{u}_0 + \epsilon \tilde{\boldsymbol{u}}) = \boldsymbol{A}(\boldsymbol{u}_0) + \epsilon \mathcal{B}\, \tilde{\boldsymbol{u}} + \frac{1}{2}\epsilon^2 \mathcal{C} \tilde{\boldsymbol{u}} \tilde{\boldsymbol{u}} + \mathcal{O}(\epsilon^3) \quad (3.4)$$

where $\mathcal{B}\, \tilde{\boldsymbol{u}} \equiv \nabla_{\boldsymbol{u}} \left(\boldsymbol{A}(\boldsymbol{u}) \tilde{\boldsymbol{u}} \right)\Big|_{\boldsymbol{u}=\boldsymbol{u_0}}$, $\mathcal{C}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}} \equiv \nabla_{\boldsymbol{u}} (\nabla_{\boldsymbol{u}} \left(\boldsymbol{A}(\boldsymbol{u}) \tilde{\boldsymbol{u}} \right) \tilde{\boldsymbol{u}})\Big|_{\boldsymbol{u}=\boldsymbol{u_0}}$.

2) Strict hyperbolicity at $\boldsymbol{u}_0$:
Jacobian matrix $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{u}_0)$ has real and distinct eigenvalues λ_j such that

$$(\boldsymbol{A} - \lambda_j \boldsymbol{I})\, \boldsymbol{r_j} = \boldsymbol{0}, \qquad \boldsymbol{l_j}\, (\boldsymbol{A} - \lambda_j \boldsymbol{I}) = \boldsymbol{0}, \qquad \boldsymbol{l_i} \cdot \boldsymbol{r_j} = \delta_{ij}, \quad (3.5)$$

$\boldsymbol{r_j}$ and $\boldsymbol{l_j}$ are the right and left eigenvectors of $\boldsymbol{A}$ corresponding to λ_j,
δ_{ij} is the Kronecker's delta.

Inserting (3.3) into the system in (3.2), applying Taylor expansion (3.4) of $\boldsymbol{A}(\boldsymbol{u})$ around $\boldsymbol{u_0}$, using multiple scale analysis, and collecting like powers of ϵ, we obtain[3]

$$\begin{aligned} \boldsymbol{u}^\epsilon_{,t} + \boldsymbol{A}(\boldsymbol{u}^\epsilon)\boldsymbol{u}^\epsilon_{,x} = {} & \epsilon^0\, (\boldsymbol{A} - \lambda_j \boldsymbol{I})\, \boldsymbol{u}_{1,\eta} + \\ & \epsilon\, (\, (\boldsymbol{A} - \lambda_j \boldsymbol{I})\, \boldsymbol{u}_{2,\eta} + \mathcal{B}\, \boldsymbol{u}_1\, \boldsymbol{u}_{1,\eta} + \\ & \boldsymbol{u}_{1,t} + \boldsymbol{A}\, \boldsymbol{u}_{1,x})\; + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.6)$$

Equating the consecutive terms to zero we get

- $\mathcal{O}(\epsilon^0)$ terms vanish $\quad \Leftrightarrow \quad (\boldsymbol{A} - \lambda_j \boldsymbol{I})\, \boldsymbol{u}_{1,\eta} = \boldsymbol{0}. \quad (\star)$

[2] Actually here we need only the first two terms at the right hand side of (3.2), the next term will be needed in the modified cases (see Sec. 3.4).

[3] By the subscript "," we denote here a partial derivative with respect to the corresponding variable which follows the subscript.

The single wave solution of this equation is

$$\boldsymbol{u}_1(t,x,\eta) = \sigma_j(t,x,\eta)\,\boldsymbol{r}_j$$

with σ_j an unknown amplitude and $\boldsymbol{r}_j$ the eigenvector of the matrix $\boldsymbol{A}$.

- $\mathcal{O}(\epsilon)$ terms vanish $\quad\Leftrightarrow\quad (\boldsymbol{A} - \lambda_j \boldsymbol{I})\,\boldsymbol{u}_{2,\eta} + \boldsymbol{\mathcal{F}} = \boldsymbol{0} \qquad (\star\star)$

with

$$\boldsymbol{\mathcal{F}} \equiv \boldsymbol{\mathcal{B}}\,\boldsymbol{u}_1\,\boldsymbol{u}_{1,\eta} + \boldsymbol{u}_{1,t} + \boldsymbol{A}\,\boldsymbol{u}_{1,x}.$$

The solvability condition $\boldsymbol{l}_j \cdot \boldsymbol{\mathcal{F}} = 0$ gives the transport evolution equations for the wave amplitudes σ_j:

$$\frac{\partial \sigma_j}{\partial t} + \lambda_j \frac{\partial \sigma_j}{\partial x} + \frac{1}{2}\Gamma_j \frac{\partial \sigma_j^2}{\partial \eta} = 0 \tag{3.7}$$

with *self interaction coefficients*

$$\Gamma_j(\boldsymbol{u}_0) = \boldsymbol{l}_j \cdot (\nabla_{\boldsymbol{u}}\,\boldsymbol{A}(\boldsymbol{u})\,\boldsymbol{r}_j)\,\boldsymbol{r}_j\Big|_{\boldsymbol{u}=\boldsymbol{u}_0}. \tag{3.8}$$

In order for the expansion to be valid, we assume also the *sublinear growth condition*:

$$|\boldsymbol{u}_2(t,x,\eta)| = o(|\eta|) \quad |\eta| \to \infty$$

Under this assumption $\boldsymbol{u}_1(t,x,\eta)$ and *not* $\boldsymbol{u}_2(t,x,\eta)$ defines the leading order asymptotics.

Remark 3.1. Let $k = 1,2,3,...$ be a natural number. We will demonstrate how to transform the partial differential equation

$$\frac{\partial \sigma}{\partial t} + \lambda \frac{\partial \sigma}{\partial x} + \frac{1}{k}\Gamma \frac{\partial \sigma^k}{\partial \eta} = 0 \tag{3.9}$$

with the function of three independent variables $\sigma = \sigma(t,x,\eta)$ to the PDE:

$$\frac{\partial \tilde{\sigma}}{\partial \tau} + \frac{1}{k}\Gamma \frac{\partial \tilde{\sigma}^k}{\partial y} = 0 \tag{3.10}$$

with the function of two independent variables $\tilde{\sigma} = \tilde{\sigma}(\tau, y)$. This transformation can be accomplished e.g. by a simple change of the independent variables:

$$\tau = x + (1-\lambda)t,$$
$$y = \eta.$$

Then we have

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \tilde{\sigma}}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial \tilde{\sigma}}{\partial \tau}(1-\lambda),$$
$$\frac{\partial \sigma}{\partial x} = \frac{\partial \tilde{\sigma}}{\partial \tau}\frac{\partial \tau}{\partial x} = \frac{\partial \tilde{\sigma}}{\partial \tau},$$
$$\frac{\partial \sigma}{\partial \eta} = \frac{\partial \tilde{\sigma}}{\partial y}\frac{\partial y}{\partial \eta} = \frac{\partial \tilde{\sigma}}{\partial y}.$$

Therefore

$$\frac{\partial \sigma}{\partial \tau} + \lambda\frac{\partial \sigma}{\partial x} + \frac{1}{k}\Gamma\frac{\partial \sigma^k}{\partial y} = \frac{\partial \tilde{\sigma}}{\partial \tau} + \frac{1}{k}\Gamma\frac{\partial \tilde{\sigma}^k}{\partial y} = 0.$$

When $k = 2$ we call the equation 3.10 the *inviscid Burgers equation*. When $k = 3$ we call the equation 3.10 the *modified inviscid Burgers equation.*

Remark 3.2. For a mode which is locally linearly degenerate at $\boldsymbol{u}_0$ the coefficient $\Gamma_j = 0$, and hence the equation (3.7) becomes linear. This is a typical situation for transverse (shear) waves. The modified expansion which gives the nonlinear evolution equations in such cases will be presented in Sec. 3.4.

Although in the process of formal derivation of the asymptotic *WNGO* equations we require smoothness, it turns out that the asymptotics is valid even after a shock formation. The classical result of DiPerna and Majda [39] states the following:

Theorem 3.3. *Let $\boldsymbol{u}^\epsilon(t,x)$ be a weak solution of the Cauchy problem for a genuine nonlinear and strictly hyperbolic system of conservation laws*

$$\begin{cases} \boldsymbol{u}^\epsilon_t + \boldsymbol{f}(\boldsymbol{u}^\epsilon)_x = \boldsymbol{0} \\ \boldsymbol{u}^\epsilon(0,x) = \boldsymbol{u}_0 + \epsilon\, \boldsymbol{v}_0(x) \end{cases}$$

with $\boldsymbol{v}_0(x)$ of bounded variation and compact support, and let

$$\boldsymbol{u}^\epsilon_{wngo}(t,x) = \boldsymbol{u}_0 + \epsilon \sum_j \sigma_j(t,x,\frac{x-\lambda_j t}{\epsilon})\, \boldsymbol{r_j}$$

be a WNGO approximation. Here λ_j *and* $\boldsymbol{r_j}$ *are the eigenvalues and the eigenvectors of the matrix* $\nabla_u \boldsymbol{f} \equiv \boldsymbol{A}$ *and* σ_j *satisfy (3.7) with (3.8).*

Then

$$\max_{0\le t<\infty} |\boldsymbol{u}^\epsilon(t,\cdot) - \boldsymbol{u}^\epsilon_{wngo}(t,\cdot)|_1 \le C\epsilon^2$$

uniformly for all times. Here the constant C *is independent of* ϵ*, and* $|\cdot|_1$ *denotes the* L^1 *norm.*

This theorem was extended by K. Zumbrun [178] to the nonconvex systems[4] with the use of modified asymptotics. The idea of modified asymptotics will be presented in Sec. 3.4.

3.3 Umbilic Points

Now we present the modifications of a classical *WNGO* expansion to handle the case of multiple eigenvalues. Suppose then the hyperbolic system (3.2) is *not strictly hyperbolic* at $\boldsymbol{u} = \boldsymbol{u}_0$. Such a point $\boldsymbol{u}_0$ at which the eigenvalues collide, is called an *umbilic point*.

3.3.1 Double Umbilic Point

Assume first that $\lambda_1(\boldsymbol{u}_0) = \lambda_2(\boldsymbol{u}_0) \equiv \lambda$ is a double eigenvalue of the matrix $\boldsymbol{A}$ from (3.2). Suppose $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$, and $\{\boldsymbol{l}_1, \boldsymbol{l}_2\}$ are corresponding right and left eigenvectors, respectively. We postulate the asymptotic expansion of the form

$$\boldsymbol{u}^\epsilon(t,x) = \epsilon\left(\sigma_1(t,x,\frac{\phi_1}{\epsilon})\boldsymbol{r}_1 + \sigma_2(t,x,\frac{\phi_2}{\epsilon})\,\boldsymbol{r}_2\right), \tag{3.11}$$

where $\phi_1 = x - \lambda_1 t, \phi_2 = x - \lambda_2 t$. Plugging such expansion into the system (3.2), using multiple scale analysis, collecting terms with similar powers of ϵ, and then equating the consecutive terms to zero, we obtain the equation ($\star$). The solution to the equation ($\star$) consists now of two terms:

$$\boldsymbol{u}_1(t,x,\eta) = \sigma_1(t,x,\eta)\,\boldsymbol{r}_1 + \sigma_2(t,x,\eta)\,\boldsymbol{r}_2 \tag{3.12}$$

with σ_1, σ_2 the unknown amplitudes, $\boldsymbol{r}_1, \boldsymbol{r}_2$ the eigenvectors of the matrix $\boldsymbol{A}$ and $\eta = \dfrac{x-\lambda t}{\epsilon}$.

[4] In [178] Zumbrun actually proved his modification of DiPerna-Majda theorem for the nonconvex p-system, but possibilities of some extensions to more general systems are discussed in his dissertation [179], yet the proof for general nonconvex systems is still missing.

Then equating to zero the next order terms we arrive at the equation

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{u}_{2,\eta} + \boldsymbol{\mathcal{F}} = \boldsymbol{0} \qquad (\star\star)$$

with

$$\boldsymbol{\mathcal{F}} \equiv \mathcal{B}\, \boldsymbol{u}_1\, \boldsymbol{u}_{1,\eta} + \boldsymbol{u}_{1,t} + \boldsymbol{A}\, \boldsymbol{u}_{1,x}.$$

Writting $\boldsymbol{\mathcal{F}}$ with the use of (3.12), and imposing the solvability condition we have

$$\begin{aligned}\boldsymbol{l}_1 \cdot \boldsymbol{\mathcal{F}} = \boldsymbol{l}_1 \cdot (&\mathcal{B}\boldsymbol{r}_1\boldsymbol{r}_1\sigma_1\sigma_{1,\eta} + \mathcal{B}\boldsymbol{r}_1\boldsymbol{r}_2\sigma_1\sigma_{2,\eta} + \mathcal{B}\boldsymbol{r}_2\boldsymbol{r}_1\sigma_2\sigma_{1,\eta} + \mathcal{B}\boldsymbol{r}_2\boldsymbol{r}_2\sigma_2\sigma_{2,\eta}) + \\ &\sigma_{1,t} + \lambda\sigma_{1,x} = 0,\end{aligned}$$

$$\begin{aligned}\boldsymbol{l}_2 \cdot \boldsymbol{\mathcal{F}} = \boldsymbol{l}_2 \cdot (&\mathcal{B}\boldsymbol{r}_1\boldsymbol{r}_1\sigma_1\sigma_{1,\eta} + \mathcal{B}\boldsymbol{r}_1\boldsymbol{r}_2\sigma_1\sigma_{2,\eta} + \mathcal{B}\boldsymbol{r}_2\boldsymbol{r}_1\sigma_2\sigma_{1,\eta} + \mathcal{B}\boldsymbol{r}_2\boldsymbol{r}_2\sigma_2\sigma_{2,\eta}) + \\ &\sigma_{2,t} + \lambda\sigma_{2,x} = 0,\end{aligned}$$

where we recall that $\mathcal{B} = \mathcal{B}(\boldsymbol{u}_0) = (\nabla_{\boldsymbol{u}}\, \boldsymbol{A})\big|_{\boldsymbol{u}=\boldsymbol{u}_0}$.

Hence the evolution equations for the waves amplitudes look as follows

$$\frac{\partial \sigma_1}{\partial t} + \lambda \frac{\partial \sigma_1}{\partial x} + \Gamma^1_{11}\sigma_1 \frac{\partial \sigma_1}{\partial \eta} + \Gamma^1_{21}\sigma_2 \frac{\partial \sigma_1}{\partial \eta} + \Gamma^1_{12}\sigma_1 \frac{\partial \sigma_2}{\partial \eta} + \Gamma^1_{22}\sigma_2 \frac{\partial \sigma_2}{\partial \eta} = 0, \tag{3.13}$$

$$\frac{\partial \sigma_2}{\partial t} + \lambda \frac{\partial \sigma_2}{\partial x} + \Gamma^2_{11}\sigma_1 \frac{\partial \sigma_1}{\partial \eta} + \Gamma^2_{21}\sigma_2 \frac{\partial \sigma_1}{\partial \eta} + \Gamma^2_{12}\sigma_1 \frac{\partial \sigma_2}{\partial \eta} + \Gamma^2_{22}\sigma_2 \frac{\partial \sigma_2}{\partial \eta} = 0, \tag{3.14}$$

where Γ^j_{pq} are defined by

$$\Gamma^j_{pq}(\boldsymbol{u}_0) = \boldsymbol{l}_j \cdot [(\nabla_{\boldsymbol{u}}\, \boldsymbol{A}(\boldsymbol{u})\, \boldsymbol{r}_p)\, \boldsymbol{r}_q]\big|_{\boldsymbol{u}=\boldsymbol{u}_0}. \tag{3.15}$$

We get the pair of coupled nonlinear PDE's which are the asymptotic evolution equations for the amplitudes of the pair of nonstrictly hyperbolic waves in the vicinity of a double umbilic point. This general form of coupled evolution equations (3.13), (3.14) may further be simplified in the presence of a particular symmetry. Such a case may happen in particular when some of the Γ^j_{pq} are zero. We would like to present here an important case of such evolution equations which arise when we have to deal with plane waves propagating along the axes of a three-fold symmetry.

We derive canonical asymptotic equations in the case which will be useful when we discuss a cubic crystal. Then the canonical evolution equations for two pairs of shear waves propagating along the $[1, 1, 1]$ direction are *complex Burgers equations*.

Lemma 3.4. *Suppose we have the* 2×2 *nonstrictly hyperbolic system of conservation laws:*

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = \boldsymbol{0}.$$

Let $\boldsymbol{u}_0$ *be the umbilic point such that* $\lambda_1(\boldsymbol{u}_0) = \lambda_2(\boldsymbol{u}_0) = \lambda$*, and at* $\boldsymbol{u}_0$ *we have*[5]

$$\Gamma^1_{11} = -\Gamma^1_{22} = -\Gamma^2_{12} = -\Gamma^2_{21} = a,$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0.$$

2	0	a
1	a	0
	1	2

2	ā	0
1	0	ā
	1	2

Fig. 3.1.

Then the asymptotic evolution equations for wave's amplitudes σ_1 *and* σ_2*, reduce to the following system:*

$$\frac{\partial \sigma_1}{\partial t} + \lambda \frac{\partial \sigma_1}{\partial x} + a\,(\sigma_1 \frac{\partial \sigma_1}{\partial \eta} - \sigma_2 \frac{\partial \sigma_2}{\partial \eta}) = 0, \qquad (3.16)$$

$$\frac{\partial \sigma_2}{\partial t} + \lambda \frac{\partial \sigma_2}{\partial x} - a\,(\sigma_1 \frac{\partial \sigma_2}{\partial \eta} + \sigma_2 \frac{\partial \sigma_1}{\partial \eta}) = 0. \qquad (3.17)$$

Remark 3.5. After a simple change of variables:

$$\tau = x + (1 - \lambda)t, \qquad \theta = \eta,$$

the system (3.16) and (3.17) becomes a *complex Burgers* system (c.f. Sec. 4.1.3):

[5] We illustrate this situation on the **Fig. 3.1**, where $\bar{a} \equiv -a$. The left table represent the values of the coefficients Γ^1_{pq}, and the right one Γ^2_{pq}. From the left table we have $\Gamma^1_{11} = -\Gamma^1_{22} = a$, and $\Gamma^1_{12} = \Gamma^1_{21} = 0$. From the right table we have $\Gamma^2_{12} = \Gamma^2_{21} = -a$, and $\Gamma^2_{11} = \Gamma^2_{22} = 0$.

$$\begin{cases} \tilde{\sigma}_{1,\tau} + \frac{a}{2}\,(\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2)_{,\theta} = 0 \\ \tilde{\sigma}_{2,\tau} - a\,(\tilde{\sigma}_1\,\tilde{\sigma}_2)_{,\theta} \quad = 0. \end{cases} \tag{3.18}$$

Some special solutions of complex Burgers equations have been studied by S. Noelle in his Ph.D. thesis [128], see also [129] and [130].

3.3.2 Multiple Eigenvalues

Now assume that there exists k eigenvalues of the matrix $\mathbf{A}$ from (3.2) such that $\lambda_1(\boldsymbol{u}_0) = ... = \lambda_k(\boldsymbol{u}_0) = \lambda$; and k right and left corresponding eigenvectors $\{\boldsymbol{r}_1, ..., \boldsymbol{r}_k\}$, and $\{\boldsymbol{l}_1, ..., \boldsymbol{l}_k\}$ respectively. We assume that the algebraic multiplicity k of such eigenvalue $\lambda_j(\boldsymbol{u}_0)$ is equal to its geometric multiplicity. As we carry out the analysis based on the asymptotic expansion (3.3), we get the solution to the equation ($\star$) given now by

$$\boldsymbol{u}_1(t, x, \eta) = \sum_{j=1}^{k} \sigma_j(t, x, \eta)\; \boldsymbol{r}_j,$$

where σ_j are the unknown amplitudes, and $\eta = \epsilon^{-1}(x - \lambda_j t)$. Moreover, λ_j and $\boldsymbol{r}_j$ represent the wave speeds and polarizations of the j-th wave respectively. The solvability condition to the ($\star\star$) equation gives now k coupled equations for the unknown amplitudes:

$$\frac{\partial \sigma_j}{\partial t} + \lambda_j \frac{\partial \sigma_j}{\partial x} + \Gamma^j_{jj}\,\sigma_j\,\frac{\partial \sigma_j}{\partial \eta} + \frac{1}{2}\sum_{p,q} \Gamma^j_{pq} \sigma_p \frac{\partial \sigma_q}{\partial \eta} = 0 \tag{3.19}$$

for j = 1,..., k, and p < q. Here $\Gamma^j_{jj} \equiv \Gamma_j$ are the *self interaction coefficients* (3.8), while Γ^j_{pq} for $j \neq p \neq q$ are the *three waves interaction coefficients* which are defined as

$$\Gamma^j_{pq}(\boldsymbol{u}_0) = \boldsymbol{l}_j \cdot [(\nabla_{\boldsymbol{u}}\, \boldsymbol{A}(\boldsymbol{u})\, \boldsymbol{r}_p)\, \boldsymbol{r}_q]\Big|_{\boldsymbol{u}=\boldsymbol{u}_0}. \tag{3.20}$$

The interaction coefficients measure the strength of 'p' and 'q' waves interaction on the 'j' wave.

3.4 Modified Asymptotics

Here we concentrate on modifications of the classical *WNGO* expansion presented in the previous sections, to the cases in which we encounter a *local loss of genuine nonlinearity.* Let us assume that $\lambda_s(\boldsymbol{u})$ is an eigenvalue of the matrix $\boldsymbol{A}(\boldsymbol{u})$ from (3.2) with the corresponding eigenvector $\boldsymbol{r}_s$ such that

$$(\nabla_{\boldsymbol{u}} \lambda_s(\boldsymbol{u}) \cdot \boldsymbol{r}_s)\Big|_{\boldsymbol{u}=\boldsymbol{0}} = 0.$$

We begin with a formal expansion of the solution to the Cauchy problem (3.2), similar to the expansion (3.3). This time, however, we take into account one more term in the expansion, and we consider a different scaling

$$\boldsymbol{u}^\epsilon(t,x) = \boldsymbol{u}_0 + \epsilon \boldsymbol{u}_1(t,x,\eta) + \epsilon^2 \boldsymbol{u}_2(t,x,\eta) + \epsilon^3 \boldsymbol{u}_3(t,x,\eta) + \mathcal{O}(\epsilon^4) \quad (3.21)$$

where now $\eta = \epsilon^{-2}(x - \lambda_s t)$. Such a scaling is typical for cubic nonlinearities which are characteristic for shear waves.
Next we insert the expansion (3.21) into the equations (3.2) and proceed in an analogous way as in the previous section. As a result we obtain the following equation (on the right hand side of the equality the terms are grouped in increasing powers of ϵ):

$$\begin{aligned} \boldsymbol{u}^\epsilon_{,t} + \boldsymbol{A}(\boldsymbol{u}^\epsilon)\boldsymbol{u}^\epsilon_{,x} = {} & \epsilon^{-1}(\boldsymbol{A} - \lambda_s \boldsymbol{I})\, \boldsymbol{u}_{1,\eta} + \\ & \epsilon^0 \; (\, (\boldsymbol{A} - \lambda_s \boldsymbol{I})\, \boldsymbol{u}_{2,\eta} + \mathcal{B}\, \boldsymbol{u}_1\, \boldsymbol{u}_{1,\eta}\,) + \\ & \epsilon \;\; (\, (\boldsymbol{A} - \lambda_s \boldsymbol{I})\, \boldsymbol{u}_{3,\eta} + (\mathcal{B}\, \boldsymbol{u}_1\, \boldsymbol{u}_2)_{,\eta} + \\ & \quad \boldsymbol{u}_{1,t} + \boldsymbol{A}\, \boldsymbol{u}_{1,x} + \tfrac{1}{6}(\mathcal{C}\boldsymbol{u}_1\boldsymbol{u}_1\boldsymbol{u}_1)_{,\eta}\,) + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.22)$$

Similarly to the previous section, we have

- $\mathcal{O}(\epsilon^{-1})$ terms vanish $\quad\Leftrightarrow\quad$ $(\boldsymbol{A} - \lambda_s \boldsymbol{I})\, \boldsymbol{u}_{1,\eta} = \boldsymbol{0}$.

Hence the single wave solution of this equation is

$$\boldsymbol{u}_1(t,x,\eta) = \sigma_s(t,x,\eta)\, \boldsymbol{r}_s$$

with $\boldsymbol{r}_s$ the eigenvector of the matrix $\boldsymbol{A}$ and σ_s an unknown function. Next

- $\mathcal{O}(\epsilon^0)$ terms vanish $\quad\Leftrightarrow\quad$ $(\boldsymbol{A} - \lambda_s \boldsymbol{I})\, \boldsymbol{u}_{2,\eta} + \mathcal{B}\, \boldsymbol{u}_1\, \boldsymbol{u}_{1,\eta} = \boldsymbol{0}$.

The solution of this equation is

$$\boldsymbol{u}_2(t,x,\eta) = \frac{1}{2}\sigma_s^2(t,x,\eta)\, \boldsymbol{q} + b(t,x,\eta)\, \boldsymbol{r}_s$$

where b is an unknown function, $\boldsymbol{r}_s$ is as before and the vector $\boldsymbol{q}$ satisfies

$$(\boldsymbol{A} - \lambda_s \boldsymbol{I})\, \boldsymbol{q} + \mathcal{B}\, \boldsymbol{r}_s\, \boldsymbol{r}_s = \boldsymbol{0}. \tag{3.23}$$

Finally

- $\mathcal{O}(\epsilon^1)$ terms vanish $\quad\Leftrightarrow\quad (\boldsymbol{A} - \lambda_s \boldsymbol{I})\, \boldsymbol{u}_{3,\eta} + \boldsymbol{\mathcal{H}} = \boldsymbol{0}$

where

$$\boldsymbol{\mathcal{H}} \equiv \boldsymbol{u}_{1,t} + \boldsymbol{A}\, \boldsymbol{u}_{1,x} + (\mathcal{B}\, \boldsymbol{u}_1\, \boldsymbol{u}_2)_{,\eta} + \frac{1}{6}(\mathcal{C}\boldsymbol{u}_1 \boldsymbol{u}_1 \boldsymbol{u}_1)_{,\eta}\, .$$

The solvability condition $\boldsymbol{l}_s \cdot \boldsymbol{\mathcal{H}} = 0$ gives the evolution equations for the amplitudes of shear waves

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \frac{1}{3} G_s \frac{\partial {\sigma_s}^3}{\partial \eta} = 0 \tag{3.24}$$

where

$$G_s = \frac{1}{2} \boldsymbol{l}_s \cdot \left(3\mathcal{B}\, \boldsymbol{r}_s\, \boldsymbol{q} + \mathcal{C}\, \boldsymbol{r}_s\, \boldsymbol{r}_s\, \boldsymbol{r}_s\right), \tag{3.25}$$

and the vector $\boldsymbol{q}$ satisfies (3.23).

3.5 Resonant Interactions

So far we have discussed the asymptotic expansions for the *IVP* with initial data of compact support. This assumption prevents waves from resonant interactions. However when the data are periodic, the system consists of at least three equations, and provided certain relations between the phases are satisfied, a nonlinear resonance may take place. In such a case new waves are produced, which propagate with frequencies being linear combinations of the frequencies of the interacting waves.

A resonant wave interaction occurs in a variety of physical systems like e.g. in gas and fluid as well as solid dynamics, plasma physics, optics etc. During resonant wave interactions the energy is exchanged between the nonlinear modes. The most investigated are the resonant quadratic nonlinear interactions of three waves. These three-wave equations are obtained as the asymptotic equations for the waves amplitudes.

The two simplest examples of the three-wave resonant interactions are *decay interactions* or *explosive interactions*. In the former case the solutions stay bounded in time while in the latter case the solutions may blow up in finite time.

Suppose there are two waves with frequencies and wave numbers (ω_1, k_1) and (ω_2, k_2) which interact resonantly and as a result they produce a third wave with a frequency and wave number (ω_3, k_3) being a linear combinations of the previous waves. The simplest resonant conditions for three waves (satisfying the relations $\omega_j = \lambda_j k_j$ for $j = 1, 2, 3$) are the following:

for decay interactions, one of the possibilities is

$$\omega_1 = \omega_2 + \omega_3$$
$$k_1 = k_2 + k_3.$$

For explosive interactions, we have

$$\omega_1 + \omega_2 + \omega_3 = 0$$
$$k_1 + k_2 + k_3 = 0.$$

Mathematically, resonant interactions manifest themselves in the presence of integrodifferential terms in the evolution transport equations for the unknown amplitudes a_j. The transport evolution equations in the resonant case have the form [118]:

$$\frac{\partial a_j}{\partial t} + \lambda_j \frac{\partial a_j}{\partial x} + \Gamma^j_{jj}\, a_j\, \frac{\partial a_j}{\partial \eta} + \tag{3.26}$$
$$\sum_{p,q}{}' \, \Gamma^j_{pq} \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \frac{\partial a_p}{\partial \eta}(t, x, \theta_{jp})\, a_q(t, x, \theta_{jq})\, ds \;= 0,$$

where $\theta_{jp} = \eta + s(\lambda_j - \lambda_p)$, $\eta \equiv (x - \lambda_j t)\, \epsilon^{-1}$, and where $\sum'$ indicates summation avoiding repeated indices.

The linear terms in (3.26) represent transport with speed λ_j, while the first nonlinear term is a Burgers' term, which causes steepening of waves and subsequent shock formation and decay. The last term is an interaction term, representing the total amount of j-wave produced by the resonant interaction of waves of the p-th and q-th families. It is clear that the coefficients of these terms measure the relative strength of nonlinear effects, and these are exactly the interaction coefficients. The signs of the interaction coefficients are important. Namely, when all interaction coefficients have the same sign, the modulation asymptotic resonant evolution equations can have solutions which blow up in finite time.

Remark 3.6. Here we have assumed that the amplitudes $a_j(t,x,\eta)$ are smooth, periodic functions of η with zero mean, i.e.

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a_j(t,x,\eta)d\eta = 0.$$

3.6 Interaction Coefficients

The interaction coefficients measure which waves interact and how strong the interaction is. In general Γ^j_{pq} may be asymmetric. We show this below but first let us formulate the following useful lemma:

Lemma 3.7. *Suppose $\lambda_j(\boldsymbol{u})$ is an eigenvalue, and $\boldsymbol{l_j}(\boldsymbol{u})$ and $\boldsymbol{r_j}(\boldsymbol{u})$ are correspondingly left and right eigenvectors of the matrix $\boldsymbol{A}(\boldsymbol{u})$ such that $\boldsymbol{l_j}(\boldsymbol{u})\cdot\boldsymbol{r_j}(\boldsymbol{u}) = 1$. Then*

$$\boldsymbol{l_j}(\boldsymbol{u})\cdot\Big(\nabla_{\boldsymbol{u}}\,\boldsymbol{A}(\boldsymbol{u})\Big)\,\boldsymbol{r_j}(\boldsymbol{u})\boldsymbol{r_q}(\boldsymbol{u}) = \nabla_{\boldsymbol{u}}\,\lambda_j(\boldsymbol{u})\cdot\boldsymbol{r_q}(\boldsymbol{u}). \tag{3.27}$$

where $\boldsymbol{r_q}(\boldsymbol{u})$ is another eigenvector of the matrix $\boldsymbol{A}(\boldsymbol{u})$.

The proof of (3.27) can be found e.g. in the book of C. Dafermos [35].

Corollary 3.8. *From (3.27) we have*

$$\Gamma^j_{jq} = \nabla_{\boldsymbol{u}}\,\lambda_j(\boldsymbol{u})\cdot\boldsymbol{r_q}(\boldsymbol{u})\Big|_{\boldsymbol{u}=\boldsymbol{u_0}},$$

in particular the self - interaction coefficient

$$\Gamma^j_{jj} = \nabla_{\boldsymbol{u}}\,\lambda_j(\boldsymbol{u})\cdot\boldsymbol{r_j}(\boldsymbol{u})\Big|_{\boldsymbol{u}=\boldsymbol{u_0}}.$$

Next we show on a simple example that the interacting coefficients can be in general nonsymmetric.

Example 3.9. Let $\mathbf{l}_j = [l_{j1}, l_{j2}]$, $\mathbf{r}_p = [r_{p1}, r_{p2}]$, $\mathbf{r}_q = [r_{q1}, r_{q2}]$, and

$$\boldsymbol{A}(\mathbf{u}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then we have

$$\begin{aligned}\Gamma^j_{pq} &= l_{j1}r_{p1}r_{q1}a_{11,1} + l_{j1}r_{p1}r_{q2}a_{11,2} + l_{j1}r_{p2}r_{q1}a_{12,1} + l_{j1}r_{p2}r_{q2}a_{12,2} + \\ &+ l_{j2}r_{p1}r_{q1}a_{21,1} + l_{j2}r_{p1}r_{q2}a_{21,2} + l_{j2}r_{p2}r_{q1}a_{22,1} + l_{j2}r_{p2}r_{q2}a_{22,2}.\end{aligned}$$

On the other hand

$$\begin{aligned}\Gamma^j_{qp} &= l_{j1}r_{p1}r_{q1}a_{11,1} + l_{j1}r_{p2}r_{q1}a_{11,2} + l_{j1}r_{p1}r_{q2}a_{12,1} + l_{j1}r_{p2}r_{q2}a_{12,2} \\ &+ l_{j2}r_{p1}r_{q1}a_{21,1} + l_{j2}r_{p2}r_{q1}a_{21,2} + l_{j2}r_{p1}r_{q2}a_{22,1} + l_{j2}r_{p2}r_{q2}a_{22,2}.\end{aligned}$$

Since

$$\Gamma^j_{pq} - \Gamma^j_{qp} = -\left(r_{p2}r_{q1} - r_{p1}r_{q2}\right)\left(l_{j1}\left(a_{11,2} - a_{12,1}\right) + l_{j2}\left(a_{21,2} - a_{22,1}\right)\right),$$

hence

$$\Gamma^j_{pq} - \Gamma^j_{qp} \neq 0.$$

However, if $\mathbf{A}(\mathbf{u})$ is a gradient of some vector $\mathbf{f}(\mathbf{u})$, that is if $a_{11} = f_{1,1}$, $a_{12} = f_{1,2}$, $a_{21} = f_{2,1}$, $a_{22} = f_{2,2}$, then obviously it is easily visible from the formula above that $\Gamma^j_{pq} - \Gamma^j_{qp} = 0$. In such a case $a_{11,2} - a_{12,1} = f_{1,12} - f_{1,21} = 0$, and $a_{21,2} - a_{22,1} = f_{2,12} - f_{2,21} = 0$. So if the quasilinear hyperbolic system comes from the system of conservation laws, then the interaction coefficients are symmetric. We will clearly see this in the part devoted to applications in elastodynamics.

4

Canonical Evolution Equations

Mathematical models of real physical phenomena lead typically to very complicated systems of partial differential equations. Using asymptotic methods, like the one presented in the previous chapter, it is possible to reduce these complicated systems to much simpler universal canonical equations. In what follows we have listed a few of the most important canonical evolution equations. Some of them turn out to be integrable.

Now we will discuss basic evolution equations, most of which arise in the process of the asymptotic approach to the modelling of weakly nonlinear hyperbolic, dissipative or dispersive waves. Typically the canonical asymptotic equations appear as a solvability condition which guarantees the uniform boundness of the main approximation. We present these canonical evolution equations in their simplest forms with all coefficients normalized to identity for simplicity. The concrete values of these coefficients may turn out to be very important in applications to particular physical problems and later on will be specified while discussing the examples from continuum mechanics.

4.1 Hyperbolic Waves

Hyperbolic equations represent a mathematical idealization of a real physical process. They describe pure dynamics (convection) of such a process. As we have seen in Chapter 2, hyperbolicity may be characterized by the requirement that the roots of the algebraic characteristic equation (corresponding to the hyperbolic differential operator) are all real. Speaking in other words we may say that hyperbolic waves propagate with a *finite* *speed.* Below we will present the simplest canonical evolution equations characterizing propagation of nonlinear hyperbolic waves.

4.1.1 Inviscid Burgers Equation

The inviscid Burgers equation or as others prefer to call it the dispersionless KdV equation[1] or else the Hopf equation, is the simplest and one of the most fundamental nonlinear evolution equations. The inviscid Burgers equation models the *longitudinal* wave motion. In a longitudinal wave the particle displacement is parallel to the direction of wave propagation. Typical examples of such waves are e.g. sound waves in air. Longitudinal waves, regardless of the nature of the medium in which they propagate, are characterized mathematically as *genuinely nonlinear* (see Chapter 2). The typical feature of such waves is that the evolution equations describing propagation of the longitudinal waves amplitudes are characterized by the convex flux, here represented by *quadratic nonlinearity.* The simplest conservative form of such equations is represented by the inviscid Burgers equation:

$$u_t + (\frac{u^2}{2})_x = 0. \tag{4.1}$$

where u is a function of time t and space x variables, and the subscripts, as usual, denote differentiation with respect to the appropriate variables. The quasilinear form of this equation looks as follows:

$$u_t + uu_x = 0. \tag{4.2}$$

Qualitative features of a solution.

The presence of the quadratic nonlinear term in the above equation (4.1) causes drastic changes in the qualitative behavior of its solution in comparison with the solution of a linear, so called, baby wave equation : $u_t + u_x = 0$.

As we have already mentioned before, the typical feature of a nonlinear hyperbolic wave motion is that even when we start with smooth data we

[1] Both names emphasize that there is no dispersion or viscosity in the process modelled by this equation.

may end up with a discontinuity. This is caused by the fact that due to the presence of the nonlinear term, the higher harmonics are travelling faster than the lower ones and force the wave to brake. In other words the nonlinear convection makes the characteristics cross each other, and this implies that the space derivatives become infinite in finite time. This is connected with the steepening of the wave's profile. Formation of shock waves is one of the most typical of such phenomena.

Lemma 4.1. *The initial value problem for inviscid Burgers equation*

$$\begin{cases} u_t + uu_x = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{4.3}$$

has a global classical solution if $u_0'(x)$ is greater or equal to zero for all real x, whereas any C^1 data cease to be continuous after some time if $u_0'(x)$ is less than zero for some x.

Proof:
The solution of (4.3) is given by the implicit formula:

$$u = u_0(x - ut). \tag{4.4}$$

We have

$$u_x = \frac{u_0'(\xi)}{1 + u_0'(\xi)t} \tag{4.5}$$

$$u_t = -\frac{uu_0'(\xi)}{1 + u_0'(\xi)t} \tag{4.6}$$

where the characteristic $\xi = x - tu$. A smooth (at least C^1) solution to the *IVP* fails to exist at any value of t for which the denominator in (4.5) or (4.6) vanishes and the numerator does not. Therefore e.g. $u_x \to \infty$ at the critical time $t_{cr} > 0$ if at some point $u_0'(\xi) < 0$. In such cases a shock wave is formed.

The large time behavior of solutions to the Cauchy problem (4.3) depends on the type of the initial data. If they are of compact support then an N-wave profile (having a shape of a letter N) is being generated after a sufficiently long time. On the other hand periodic initial data imply formation of sawtooth profiles after some time.

4.1.2 Modified Inviscid Burgers Equation

Another fundamental class of mechanical waves, apart from longitudinal, are transverse waves. In a transverse wave the particle displacement is perpendicular to the direction of wave propagation. This happens e.g. for waves propagating in an elastic string. Conservation laws with non-convex fluxes models shear waves propagation. A prototypical example is the one with a cubic flux function: $f(u) = u^3$ The modified inviscid Burgers equation is the canonical equation of that type:

$$u_t + (\frac{u^3}{3})_x = 0. \tag{4.7}$$

This equation models nonlinear motion of transverse waves or more generally waves with a rotational invariance property. An isotropic elastic material is an example of such a medium. The modified inviscid Burgers equation can model wave phenomena in a variety of media for example propagation of shear waves in a nonlinear elastic, viscoelastic or magnetoelastic medium or transverse waves in magnetohydrodynamics or waves in nonlinear dielectrics or else transverse waves in nonlinear magnetics. The obvious difference in comparison with the inviscid Burgers equation is that here the nonlinearity is stronger, so the solution breaks down earlier. The quasilinear form of this equation looks as follows:

$$u_t + u^2 u_x = 0. \tag{4.8}$$

The solution of the Cauchy problem for this equation with initial data u_0 as is given by the implicit formula:

$$u = u_0(x - u^2 t). \tag{4.9}$$

The analysis of the shock formation for the modified inviscid Burgers equation can be done analogously to the analysis done for the inviscid Burgers equation (see the previous section).

4.1.3 Complex Burgers Equation

This is the equation which can be represented as the following system

$$\begin{cases} u_t + \dfrac{1}{2}(u^2 - v^2)_x = 0 \\ v_t - (u\,v)_x = 0. \end{cases} \tag{4.10}$$

First let us explain why this system is called a *complex Burgers equation.* Suppose we add the first equation of (4.10) to the second one multiplied by the imaginary i. We get then

$$u_t + iv_t + \frac{1}{2}(u^2 - 2iu\,v - v^2)_x = 0. \tag{4.11}$$

Now assuming that $U \equiv u + iv$, we have that the complex conjugate squared $\overline{U}^2 = u^2 - 2iu\,v - v^2$, hence (4.10) can be written as

$$U_t + \frac{1}{2}(\overline{U}^2)_x = 0. \tag{4.12}$$

Remark 4.2. Similarly by subtracting the first equation of (4.10) to the second one multiplied by imaginary i, we get

$$\overline{U}_t + \frac{1}{2}(U^2)_x = 0. \tag{4.13}$$

This equation is also called a complex Burgers equation.

Lemma 4.3. *If $U(t,x)$ satisfies the complex Burgers equation (4.10), then so does $V(t,x) \equiv e^{\frac{2\pi i}{3}} U(t,x)$.*

Proof: We have $\overline{V}^{\,2} = e^{\frac{-4\pi i}{3}} \overline{U}^{\,2}$, so

$$V_t + \frac{1}{2}(\overline{V})^2{}_x = e^{\frac{2\pi i}{3}} U_t + \frac{1}{2} e^{\frac{-4\pi i}{3}} \overline{U}^2{}_x = e^{\frac{2\pi i}{3}} \left(U_t + \frac{1}{2}(\overline{U})^2{}_x\right) = 0.$$

Definition 4.4. *We say that the periodic function $U(t,x)$ with period L ($U(t,x) = U(t,x+L)$) has the threefold symmetry if for all t and x*

$$U(t, x + \frac{L}{3}) = e^{\frac{2\pi i}{3}} U(t,x). \tag{4.14}$$

From Lemma 4.3 and Definition 4.4 it follows that a periodic function with a three-fold symmetry satisfies (4.10). Some solutions to the complex Burgers equations (4.10) were studied by S. Noelle in his thesis [128], see also [129], [130]. These equations reveal interesting features like e.g. formation of nonclassical - undercompressive shock waves [129]. This will be more clear after we analyze the eigensystem for these equations and when we show below that a loss of genuine nonlinearity occurs for some range of parameters for the complex Burgers equations. The problem od stability of undercompressive shocks for complex Burgers equation was investigated by T. P. Liu and K. Zumbrun in [113].

Later on we will demonstrate that this model (the complex Burgers equation) is the canonical asymptotic model for the interaction of (quasi)-shear elastic plane waves propagating along the axis of the trigonal symmetry in a cubic crystal. We presented this result for the first time ([53]) at the Seventh International Conference on Hyperbolic Problems in Zürich in 1998 (see also [54], [56]). To our knowledge this is the first time the complex Burgers equations appeared as an asymptotic model in the context of elasticity.[2]

Let us now come back to the system (4.10) and analyze the structure of the eigensystem of the quasilinear form of (4.10):

$$\boldsymbol{L}\boldsymbol{w} = \frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{A}(\boldsymbol{w})\,\frac{\partial \boldsymbol{w}}{\partial x} = \boldsymbol{0}, \tag{4.15}$$

where $\boldsymbol{w} = [\,u, v\,]^T$, and

$$\boldsymbol{A}(\boldsymbol{w}) = \begin{pmatrix} u & -v \\ -v & -u \end{pmatrix}.$$

The eigenvalues are

$$\begin{aligned} \lambda_1(\boldsymbol{w}) &= -\sqrt{u^2+v^2} = -|\boldsymbol{w}|, \\ \lambda_2(\boldsymbol{w}) &= \ \sqrt{u^2+v^2} \ \ = \ \ |\boldsymbol{w}|. \end{aligned}$$

We have

$$\begin{aligned} \nabla_{\boldsymbol{w}}\,\lambda_1(\boldsymbol{w}) &= -\frac{1}{|\boldsymbol{w}|}(u, v), \\ \nabla_{\boldsymbol{w}}\,\lambda_2(\boldsymbol{w}) &= \ \frac{1}{|\boldsymbol{w}|}(u, v). \end{aligned}$$

[2] In a recent joint paper (W. Domański and A. N. Norris [65]), we generalized this fact and proved that the complex Burgers equation is a canonical asymptotic model for the interaction of (quasi)- shear elastic plane waves propagating along the axes of the three-fold symmetry in a *crystal of arbitrary anisotropy.*

The corresponding eigenvectors have the form:

$$\boldsymbol{r}_1(\boldsymbol{w}) = \left[-u + \sqrt{u^2+v^2}, v\right] = \left[-u + |\boldsymbol{w}|, v\right],$$
$$\boldsymbol{r}_2(\boldsymbol{w}) = \left[-u - \sqrt{u^2+v^2}, v\right] = \left[-u - |\boldsymbol{w}|, v\right].$$

We calculate

$$\nabla_{\boldsymbol{w}}\, \lambda_1(\boldsymbol{w}) \cdot \boldsymbol{r}_1 = \frac{u^2 - v^2 - u\,|\boldsymbol{w}|}{|\boldsymbol{w}|},$$
$$\nabla_{\boldsymbol{w}}\, \lambda_2(\boldsymbol{w}) \cdot \boldsymbol{r}_2 = \frac{v^2 - u^2 - u\,|\boldsymbol{w}|}{|\boldsymbol{w}|}.$$

One can easily check that the genuine nonlinearity is lost if $v^2 = 3u^2$ for $u < 0$ in the first family and $u > 0$ in the second family.

Remark 4.5. Although we concentrate in this work on hyperbolic waves, we will also mention very briefly other canonical models e.g. for dissipative or dispersive waves. Next section is devoted to nonlinear dissipative waves.

4.2 Dissipative Waves

This type of waves arise when a nonlinear convection is enhanced by the presence of dissipation. The basic feature of dissipation is that it smooths out the solution. Dissipative effect may be caused e.g. by heat conduction or viscosity or electrical resistivity. When the dissipation is linear the fundamental canonical evolution equation is the *Burgers equation*.

4.2.1 Burgers Equation

This is a completely integrable equation through the Cole–Hopf transform and reduction to the heat equation (see [168]). Burgers Equation has the following basic conservative form:

$$u_t + \frac{1}{2}(u^2)_x - u_{xx} = 0 \tag{4.16}$$

and the quasiliner form:

$$u_t + uu_x - u_{xx} = 0. \tag{4.17}$$

4.2.2 Modified Burgers Equation

Similarly to the inviscid case we have now dissipative equations with a nonconvex flux function and the simplest prototype of such equations is a modified Burgers equation with cubic nonlinearity. Its conservative form is:

$$u_t + (\frac{u^3}{3})_x - u_{xx} = 0, \tag{4.18}$$

and the quasiliner form looks as follows:

$$u_t + u^2 u_x - u_{xx} = 0. \tag{4.19}$$

The following section deals very briefly with dispersive waves.

4.3 Dispersive Waves

Typical features of dispersive waves are oscillations. A canonical equation describing the propagation of dispersive waves is called the *Korteweg-de Vries (KdV)* equation.

4.3.1 Korteweg–de Vries (KdV) Equation

$$u_t + (\frac{u^2}{2})_x + u_{xxx} = 0 \qquad \text{conservative form} \tag{4.20}$$

$$u_t + u u_x + u_{xxx} = 0 \qquad \text{quasilinear form} \tag{4.21}$$

KdV equations occur in several physical situations such as plasma physics [26], meteorology and more importantly in the shallow water-waves context where A KdV equation contains terms with a weak (quadratic) nonlinearity and a linear (with the third derivative) dispersion. The interactions between these two effects are balanced in such a way that a peculiar wave called a *soliton* or more generally a *solitary wave* is formed. A solitary wave represents a single isolated symmetrical hump propagating without change of its form. KdV is the first equation for which the soliton or the solitary wave solution was found and the first equation shown to be completely integrable.

Korteweg-de Vries systems are considered as asymptotical equations as the amplitude of the wave is considered small whereas the wavelength is large. A KdV equation arises as an approximate model to many physical systems like e.g. plasma physics [174] but more importantly in the shallow water waves context which is the historical background in which Korteweg and de Vries obtained their results. John Scott Russell was the one who first observed experimentally the solitary wave on the Edinburg – Glasgow Canal in 1834. In 1895 D.J. Korteweg and G. de Vries [102] proposed a mathematical model equations explaining the phenomenon observed by J. S. Russel. However it was in the papers of J. Boussinesq ([18, 19]), where this equation was first formulated.

How is a soliton formed? Well, the basic mechanism is a subtle balance between a nonlinear convection and a linear dispersion. The linear dispersion prevents the breaking phenomenon caused by the nonlinear convection by splitting the more and more steeping front into the series of impulses (oscillations). Solitons are local travelling impulses of the wave type. They behave like particles and may be characterized by elastic type collisions and propagation without changing of the initial shape (only with a phase shift).

Many features of solitons are shared by *compactons*, the solutions of a nonlinear dispersive equations (see 4.28), discovered quite recently by Ph. Rosenau and J. Hyman [142], see also Rosenau [139]. However, unlike solitons, compactons do not have exponential tails.

Solitary waves in solids can arise in different cases e.g. in microstructered materials, in rods. On the other hand perturbations to a crack front in a brittle material result in long-lived and highly localized waves ('front waves') with many of the properties of solitons.

4.3.2 MKdV Equation

Modified Korteweg-de Vries equation contains a cubic convective term as well as the third derivative dispersive terms. The conservative and quasilinear forms of this equation look respectively as follows:

$$u_t + (\frac{u^3}{3})_x + u_{xxx} = 0, \tag{4.22}$$

$$u_t + u^2 u_x + u_{xxx} = 0. \tag{4.23}$$

4.3.3 Schrödinger Equation.

Let us now assume that u is complex valued. In general for an n - dimensional space, the Schrödinger equation has $n+1$ canonical forms. For $n = 1$ it has the following two canonical forms:

$$u_t + i(\, u_{xx} + |u|^2\, u) = 0, \tag{4.24}$$

$$u_t + i(\, u_{xx} - |u|^2\, u) = 0. \tag{4.25}$$

These are fundamental equations of nonlinear optics. The first one belongs to "self-focusing" case and the other one to "self-defocusing". Modulation instability is connected with a self-focusing case. Dark and bright solitons are solutions of Schrödinger equations.

4.4 Mixed Dissipative–Dispersive Waves

These waves appear in models where apart from the nonlinear convection, both linear dissipation and dispersion are present. The canonical evolution equation is called a *KdV–Burgers* equation.

4.4.1 KdV–Burgers Equation

It has the following quasilinear form:

$$u_t + uu_x + u_{xxx} - u_{xx} = 0. \tag{4.26}$$

4.4.2 MKdV–Burgers Equation

The corresponding *modified KdV–Burgers* equation has the following canonical quasilinear form:

$$u_t + u^2 u_x + u_{xxx} - u_{xx} = 0. \tag{4.27}$$

4.5 More Nonlinearities

A mutual interaction of a nonlinear dispersion with a nonlinear convection can cause fascinating phenomena which only recently have been revealed. Here we can mention the discovery of *compactons* [139] – solitary waves with a compact support, *peakons* – solitary waves with peaks [25] or *cuspons* – solitary waves with cusps [165].

Peakons were observed in the context of shallow water equations [25] now called Camassa – Holmes equations. Compactons and cuspons on the other hand were discovered as solutions of some abstract equations. We do hope however that they will be observed in nature. Both compactons and cuspons as well as peakons are nonsmooth solitary waves free of the exponential tail. Peakons, additionally, have corners while cuspons have cusps at their crest. The common feature of all these new solutions of weakly nonlinear dispersive equtions is their *nonanaliticity.*

Nonlinear Convection and Dispersion

Compactons were discovered by Ph. Rosenau and J. Hyman [142], see also Rosenau [139] as solution of the so called K(2,2) equation. This equation contains both nonlinear dispersive as well as nonlinear convective terms. The general K(m,n) equation has the following form

$$u_t + (u^m)_x + (u^n)_{xxx} = 0. \tag{4.28}$$

On the next page we present three figures with pictures of one compacton and two peakons. The first picture shows the graph of a 1D compacton which is the solution of (4.28). The following two pictures demonstrate the graphs of 1D and 2D peakons. The figures represent the functions u of space variables at a certain fixed moment of time. The horizontal axes denote space variables. In Fig. 4.1 and Fig. 4.2 u depends only on one space variable and in Fig. 4.3 u is a function of two spatial variables.

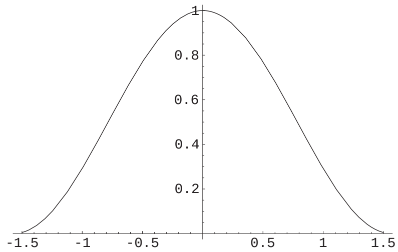

Fig. 4.1. Example of a compacton.

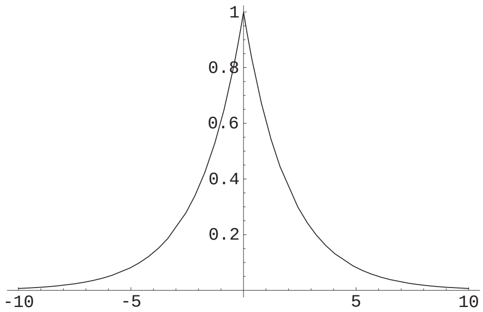

Fig. 4.2. Example of a peakon in 1-D.

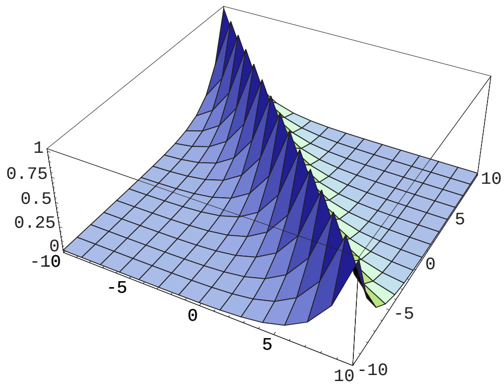

Fig. 4.3. Example of a peakon in 2-D.

Part II

APPLICATIONS TO ELASTIC SOLIDS

5

Nonlinear Elastodynamics

Introduction

In this part we study propagation and interaction of *elastic* waves in *nonlinear* media. Both *isotropic* and *anisotropic* materials are investigated. The main tools applied here, as it is in the entire work, are the perturbation methods based on weakly nonlinear asymptotics.

We begin with a general formulation of nonlinear elastodynamics written as a first order system of partial differential equations in Lagrangian coordinates in three space dimensions. The system consists of equations of motion and compatibility relations between the time derivative of the deformation gradient and the spatial gradient of the velocity vector. These equations are complemented by equations of state. We assume that the medium is *homogeneous* and *hyperelastic*. Homogeneity means no explicit dependence of the deformation gradient on the space variable. Hyperelasticity means that there exists a function (the energy density per unit volume in the reference configuration) whose derivative with respect to the deformation gradient is the first Piola–Kirchhoff stress. Both geometrical and physical nonlinearities are included in the model. We expand the strain energy up to the *third order terms* in strains. Higher order nonlinearities do not introduce new effects to our analysis. First we treat an isotropic case and then the simplest of the anisotropic medium – the most symmetrical of cubic crystals characterized by three second order and six third order elastic constants.

After introducing the constitutive relations into the equations of motion, we obtain a quasilinear first order system of partial differential equations as the modelling equations. Under certain restrictions on the energy function, our system becomes *hyperbolic*. Since we are interested in propagation of plane waves, hence we restrict ourselves to the one-space dimensional problem and obtain a 6×6 system with three components of the *velocity* vector and three components of the *displacement gradient* as the unknown dependent variables.

As the first task we are interested in global existence of smooth solutions to the Cauchy problem for a nonlinear elastodynamics system. As we have seen in the discussion in part one, for nonlinear hyperbolic equations singularities may develop in finite time even for small initial data. The null condition was introduced to prevent the blow-up and to assure a global in time existence of a classical solution to the initial value problem for the isotropic elastodynamics with small initial data [152], [2]. This condition was adopted from the general theory of nonlinear hyperbolic wave equations [100]. The null condition imposes restrictions on the quadratically nonlinear terms. We investigate this condition for the system of elastodynamics and study its connection with the property of genuine nonlinearity as well as its relation with the phenomenon of *self–resonance* of nonlinear elastic waves. We propose to use the *self–interaction coefficients* in the expression for the null condition. Using a special structure of plane waves elastodynamics [64], we formulate the null condition in a simpler way (c.f. the formula (7.28)). Moreover, we analyze this condition for a special type of elastic materials [133].

In the anisotropic case the forms of the strain energy function and hence the equations of motion change according to the chosen direction. We demonstrate the applications of *WNGO* to nonlinear elastic plane waves propagating both in the isotropic material and also along selected directions in the most symmetric of the anisotropic solids – *a cubic crystal.* The chosen directions are $[1, 0, 0], [1, 1, 0]$ and $[1, 1, 1]$. These particular directions were picked so that for each of them a clear distinction into pure longitudinal and pure transverse (shear) modes is possible for the linearized case. Moreover the choice of any of these directions illustrates a different type of *local degeneracy* with regard to strict hyperbolicity and genuine nonlinearity. Additionally we analyze also the case of an arbitrary direction in the plane $(1, 1, 0)$.

In all studied cases longitudinal waves are always *locally genuinely nonlinear* and *strictly hyperbolic* at the zero constant state. Using classical *WNGO*, we derive the inviscid Burgers equations as the canonical transport equations for all these longitudinal waves. On the other hand, due to the local *loss of genuine nonlinearity* for (quasi) shear elastic waves, applications of classical *WNGO* result in only a *linear* evolution equation for such waves. Therefore in order to obtain more adequate nonlinear evolution equations for shear waves, we have to modify a classical expansion. We consider a longer time scale and change the scaling of a small parameter. As a result using such a modified expansion, we derive uncoupled inviscid *modified Burgers equations* for locally linearly degenerate (quasi) shear elastic waves.

The only known to me previous paper devoted to applications of classical *WNGO* to nonlinear elastodynamics is due to John Hunter [83]. However he restricted himself to the use of the 4×4 system of elasticity equations in an isotropic medium, hence restricted himself to the strictly hyperbolic case and did not discuss the consequences of loss of genuine nonlinearity for shear waves. In our work we study the full 6×6 system and we apply a modified expansion ($MWNGO$) to get proper quasilinear evolution equations for transverse waves. On the contrary to Hunter's work, our main concern is to focus on cases when *strict hyperbolicity* and *genuine nonlinearity fail*. We identify when this happens and derive the asymptotic equations in such cases.

The most interesting case turned out to be the [1,1,1] direction in the cubic crystal. In this case, we derive a *new (in the context of nonlinear elasticity) coupled system* for the evolution of two shear waves, one of which is locally genuinely nonlinear and the other is locally linearly degenerate. These new evolution equations turned out to be the *complex Burgers* equations.

It is worthwhile to emphasize that we have derived general formulas for the interaction coefficients of nonlinear plane waves propagating in any direction in an arbitrary hyperelastic medium. The formulas are expressed in terms of the derivatives of the strain energy and the eigensystem of the appropriate Hessian matrix build out of the second derivative of the strain energy. We then use our formulas to calculate explicitly all the interaction coefficients in the general isotropic medium as well as in selected directions of the cubic crystal. The influence of geometrical and physical nonlinearities on the interaction coefficients is also investigated.

5.1 Fundamental Notions

Before presenting the basic equations of nonlinear elasticity, we will introduce fundamental notions needed in the description of a deformable continuum. The deformation is called *elastic* if a deformable material re-

turns to its original shape while stress is relieved. The motion of a deformable elastic continuum may be typically described using either *spatial (Eulerian)* or *material (Lagrangian)* coordinates. The distinction between these coordinates is irrelevant when we restrict ourselves to linear theories (for small - infinitesimal deformations) only. However it is important in the case of a *nonlinear* motion (for finite deformations) which we discuss in this work.[1] The material (Lagrangian) frame in which a coordinate $\boldsymbol{X}$ represents material points is usually more convenient for the analysis of deformation of a solid as opposed to spatial (Eulerian) in which a coordinate $\boldsymbol{x}$ represents a position in a physical space, more natural for the analysis of fluids. Therefore all vector and tensor fields which appear in this chapter are expressed in Lagrangian coordinates. Moreover they are assumed to be differentiable functions of space and time variables if not stated otherwise.
If $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{X})$ denotes the *displacement* in the reference configuration, then the relation at time t between the Eulerian and Lagrangian coordinates systems is expressed by the formula $\boldsymbol{u}(t, \boldsymbol{X}) = \boldsymbol{x}(t, \boldsymbol{X}) - \boldsymbol{X}$ or $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ in short .
A *deformation gradient* tensor $\boldsymbol{F} = \boldsymbol{F}(t, \boldsymbol{X}) = \nabla_{\boldsymbol{X}} \boldsymbol{x}(t, \boldsymbol{X})$, can be represented by the formula:

$$\boldsymbol{F} = \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \right).$$

We assume that $J \equiv det \boldsymbol{F} > 0$. We have $\boldsymbol{F} = \boldsymbol{I} + \nabla \boldsymbol{u}$, where $\nabla \boldsymbol{u}$ is a *displacement gradient.* The deformation gradient is uniquely split into a pure *rotation* $\boldsymbol{R}$ and a pure *stretch* $\boldsymbol{U}$:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \tag{5.1}$$

with $\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}$, and $\boldsymbol{U} = \boldsymbol{U}^T$. This is called a *polar decomposition.*

The rotational independent part of the deformation is called a *strain.* There are many possible strain measures, one is e.g. the *left Cauchy–Green strain* tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$, the other is the *right Cauchy–Green strain* tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. We have $\boldsymbol{C} = (\boldsymbol{R}\boldsymbol{U})^T \boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U} = \boldsymbol{U}^2$. Both $\boldsymbol{U}$ and $\boldsymbol{C}$ have the same orthonormal eigenvectors $\boldsymbol{N}_j$ called the *principal directions.* The eigenvalues λ_j of the symmetric tensor $\boldsymbol{U}$ are called the *principal stretches*,

[1] The comprehensive treatment of linear theory of elasticity can be found in the book of R. Hetnarski and J. Ignaczak [79].

while the eigenvalues of $\boldsymbol{C} = \boldsymbol{U}^2$ are squares of the principal stretches λ_j^2. We have

$$\boldsymbol{C} = \sum_{j=1}^{3} \lambda_j^2 \boldsymbol{N}_j \otimes \boldsymbol{N}_j.$$

The equivalent measure of strain is a *Lagrangian–Green strain* tensor

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T + \nabla\boldsymbol{u}^T\nabla\boldsymbol{u}). \tag{5.2}$$

This tensor vanishes if and only if $\boldsymbol{F}$ is orthogonal, so $\boldsymbol{E} = \boldsymbol{0}$ in the reference stress free configuration.

Deformation of the material causes changes of its internal energy which in turn implies the appearance of *stresses*. In the equations of motion for a deformable continuum, we will use the *first Piola–Kirchhoff 'engineering stress tensor'* $\boldsymbol{S}$. It represents a force acting on a unit *material* area.
The second Piola-Kirchhoff stress tensor $\boldsymbol{T}$ is defined as

$$\boldsymbol{T} = \boldsymbol{F}^{-1}\boldsymbol{S}, \quad \text{so} \quad \boldsymbol{S} = \boldsymbol{F}\boldsymbol{T}.$$

The nonsymmetric first Piola–Kirchhoff stress tensor relates forces in the current configuration to areas in the reference configuration, while the symmetric second Piola–Kirchhoff stress tensor relates forces in the reference configuration to the areas also in the reference configuration.

Remark 5.1. Just for comparison, we present the formula for the *Cauchy stress tensor* $\boldsymbol{\sigma}$ which measures stress in *spatial* coordinates. Its relation to the first Piola–Kirchhoff stress tensor $\boldsymbol{S}$ can be obtained from the Nanson's formula and looks as follows (see e.g G. A. Holzapfel [81]):

$$\boldsymbol{\sigma} = J^{-1}\, \boldsymbol{S}\boldsymbol{F}^T.$$

The transformations between material and spatial quantities are called *push–forward* and *pull–back*, the notions which are used in differential geometry (see [122] for a description of continuum mechanics with the use of the language of differential geometry).

5.2 Equations of Nonlinear Elastodynamics

The equations of elastodynamics written in Lagrangian form and in the absence of body forces, have the form

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mathrm{Div}\, \boldsymbol{S}, \tag{5.3}$$

where ρ_0 is the constant density (in the reference configuration), $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{S}$ is the first Piola-Kirchhoff stress tensor and Div denotes the divergence operator in the reference configuration.

These equations may also be written as a first-order system of partial differential equations, namely

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = \mathrm{Div}\, \boldsymbol{S}, \quad \frac{\partial \boldsymbol{F}}{\partial t} = \mathrm{Grad}\, \boldsymbol{v}, \tag{5.4}$$

where $\boldsymbol{v}$ is the velocity vector, $\boldsymbol{F}$ is the deformation gradient tensor and Grad denotes the gradient operator in the reference configuration.

In what follows we assume that the medium is *hyperelastic*. This means that there exists an energy density function $W = W(\boldsymbol{F})$, defined per unit volume in the reference configuration, such that

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}}. \tag{5.5}$$

It is assumed that W is *objective*, so that it depends on $\boldsymbol{F}$ only through the right Cauchy–Green deformation tensor $\boldsymbol{C}$, which is defined as $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$ (see, for example, [133, 134]).

Next, we define the *acoustic tensor* $\boldsymbol{Q}(\boldsymbol{N})$, which, for any vectors $\boldsymbol{n}$ (Eulerian) and $\boldsymbol{N}$ (Lagrangian), satisfies

$$\boldsymbol{n} \cdot [\boldsymbol{Q}(\boldsymbol{N})\boldsymbol{n}] = \boldsymbol{\mathcal{A}}[\boldsymbol{n} \otimes \boldsymbol{N}, \boldsymbol{n} \otimes \boldsymbol{N}] \equiv \mathcal{A}_{ijkl} n_i n_k N_j N_l,$$

where $\boldsymbol{\mathcal{A}} = \mathrm{D}^2_{\boldsymbol{F}} W(\boldsymbol{F})$ is the elasticity tensor, with components defined by

$$\mathcal{A}_{ijkl} = \frac{\partial^2 W(\boldsymbol{F})}{\partial F_{ij} \partial F_{kl}}, \tag{5.6}$$

and where $\mathrm{D}_{\boldsymbol{F}}$ represents $\partial/\partial \boldsymbol{F}$. In component form, we have

$$Q_{ik}(\boldsymbol{N}) = \mathcal{A}_{ijkl} N_j N_l = \frac{\partial^2 W(\boldsymbol{F})}{\partial F_{ij} \partial F_{kl}} N_j N_l.$$

For the considered hyperelastic material the acoustic tensor $\boldsymbol{Q}(\boldsymbol{N})$ is *symmetric*. We assume that the *strong ellipticity condition* is satisfied. This can be expressed as positive definiteness of the acoustic tensor, that is

$$\boldsymbol{n} \cdot [\boldsymbol{Q}(\boldsymbol{N})\boldsymbol{n}] > 0 \quad \forall \boldsymbol{n}, \boldsymbol{N} \neq \boldsymbol{0}. \tag{5.7}$$

The strong ellipticity condition ensures the hyperbolicity of the elastodynamic equations, which are given in the alternative forms (5.3) or (5.4).

We assume also that a strong compression implies a big increase of energy:

$$\text{if} \quad det\boldsymbol{F} \to 0^+, \quad \text{then} \quad W(\boldsymbol{F}) \to +\infty.$$

In terms of the gradient of displacement Grad $\boldsymbol{u}$, which we denote by $\boldsymbol{D}$, so that $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{D}$, where $\boldsymbol{I}$ is the identity tensor, the stress (5.5) is written

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{D}}, \tag{5.8}$$

and equations (5.4) can be rewritten as

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = \text{Div}\left(\frac{\partial W}{\partial \boldsymbol{D}}\right), \quad \frac{\partial \boldsymbol{D}}{\partial t} = \text{Grad}\,\boldsymbol{v}. \tag{5.9}$$

Now, by taking all the terms on to one side, we may write the system (5.9) in the form

$$\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{U}} = \boldsymbol{0}, \tag{5.10}$$

where the vector $\boldsymbol{\mathcal{U}}$ consists of the three components of the velocity vector $\boldsymbol{v}$ and the nine components of the displacement gradient tensor $\boldsymbol{D}$, written in an appropriate order, and $\boldsymbol{\mathcal{L}}$ is a first order quasilinear 12×12 matrix operator, which is hyperbolic under the assumption of positive definiteness of the corresponding acoustic tensor. The system (5.10) consists of 12 equations with 12 unknowns. For definiteness, we order the components of $\boldsymbol{\mathcal{U}}$ so that

$$\mathcal{U}_i = v_i, \quad i = 1, 2, 3, \quad (\mathcal{U}_4, \mathcal{U}_5, \ldots, \mathcal{U}_{11}, \mathcal{U}_{12}) = (D_{11}, D_{12}, \ldots, D_{32}, D_{33}).$$

The components of $\boldsymbol{\mathcal{L}}$ are then defined by

$$L_{ij} = -\rho_0 \delta_{ij} \frac{\partial}{\partial t}, \quad i, j \in \{1, 2, 3\},$$

$$L_{ij} = \sum_{k=1}^{3} A_{ikj} \frac{\partial}{\partial X_k}, \quad i = 1, 2, 3, \; j = 4, 5, \ldots, 12,$$

$$L_{3+k\,1} = L_{6+k\,2} = L_{9+k\,3} = \frac{\partial}{\partial X_k}, \quad k = 1, 2, 3,$$

$$L_{ii} = -\frac{\partial}{\partial t}, \quad i = 4, 5, \ldots, 12,$$

with all other $L_{ij} = 0$, where we use the representation $A_{ikp} = \mathcal{A}_{ikjl}$, with $p = (4, 5, \ldots, 12)$ corresponding to $(jl) = (11, 12, \ldots, 33)$.

5.3 Convexity Assumptions

As we have mentioned the fact that the equations of elastodynamics belong to the class of *hyperbolic* systems can be expressed mathematically in a number of alternative ways, in particular by the formula (5.7). Another assumption which leads to hyperbolicity is the *rank-one convexity* of the energy function. We discuss this and other convexity assumptions in this section. They are expressed as some restrictions on the energy density function.

Definition 5.2. *(Convexity). We say that the energy density function* $W(\boldsymbol{F})$ *is convex if the following condition is satisfied*

$$W(s\boldsymbol{F} + (1-s)\boldsymbol{G}) \leq sW(\boldsymbol{F}) + (1-s)W(\boldsymbol{G})$$

for every $s \in [0, 1]$*, and every* $\boldsymbol{F}, \boldsymbol{G} \in \mathbf{R}^{3\times 3}$*. We say that it is it strictly convex if the above inequality is sharp.*

Remark 5.3. Defining some notions with the help of the inequalities, we will use the following convention: *sharp inequality means strict property.*

The requirement of convexity of the stored energy implies good mathematical properties of the resulting equations of elastodynamics (in particular well-posedness of an initial-value problem), however this condition is far too restrictive for the description of real materials. Therefore we define now other, less restrictive types of convexity.

Definition 5.4. *(Quasi-convexity). The energy density function* $W(\boldsymbol{F})$ *is quasi–convex at* $\boldsymbol{F}$ *if*

$$\int_{\Omega} W(\boldsymbol{F})d\Omega = |\Omega|\, W(\boldsymbol{F}) \leq \int_{\Omega} W(\boldsymbol{F} + \nabla\boldsymbol{\phi})d\Omega$$

for every bounded open set $\Omega \in \mathbf{R}^3$ *and all smooth* $\boldsymbol{\phi}$ *with compact supports in* Ω*. The symbol* $|\Omega|$ *denotes the volume of* Ω*.*

Quasi-convexity could have been a desirable property, however there is no useful characterization of quasi-convex functions.

Definition 5.5. *(Poly-convexity). A function* $W : \mathbf{R}^{3\times3} \to \mathbf{R}$ *is poly-convex if there exists a convex function* $\tilde{W}$ *such that*

$$W(\boldsymbol{F}) = \tilde{W}(\boldsymbol{F}, cof\boldsymbol{F}, det\boldsymbol{F})$$

for all $\boldsymbol{F} \in \mathbf{R}^{3\times3}$.

At the end of this section we show that the energy function for a St. Venant–Kirchhoff material is *not poly-convex.*

Definition 5.6. *(Rank-one convexity). A function* $W : \mathbf{R}^{3\times3} \to \mathbf{R}$ *is rank-one convex if* W *is convex along all rank-one directions, i.e. if* W *satisfies*

$$W(s\boldsymbol{F} + (1-s)\boldsymbol{G}) \leq sW(\boldsymbol{F}) + (1-s)W(\boldsymbol{G})$$

for every $s \in [0,1]$, *and every* $\boldsymbol{F}, \boldsymbol{G}$ *with rank* $(\boldsymbol{F} - \boldsymbol{G}) = 1$.

Remark 5.7. Rank-one convexity for isotropic medium implies a common ordering of principal stretches and principal stresses which in turn implies the so called Baker-Ericksen inequalities [7].

The following relations hold true:

convexity ⇒ poly-convexity ⇒ quasi-convexity ⇒ rank-one convexity.

The direction of the implications cannot be reversed in general, however this might be the case in some special situations, e.g. rank-one convexity implies quasi-convexity for quadratic functions in 2D.

Poly-convexity and quasi-convexity are not well understood in the context of dynamics and anisotropy (see however [161]). In particular it seems to be still an open problem to characterize the poly-convexity of anisotropic media even in the case of a cubic crystal.

It is not known whether we can prove the existence of a solution for a nonlinear elastodynamics with the poly-convex or the quasi-convex energy. However, for at least C^2 energy function, the strict rank–one convexity of the sufficiently smooth energy function is equivalent to the strong ellipticity condition which guarantee hyperbolicity of the elastodynamics system. We will therefore assume that the elastic energy density function is strictly rank–one convex.

5.4 Fréchet Derivative

We recall here for completeness the definition of a Fréchet derivative, although we have used it already before. Intuitively, the Fréchet derivative generalizes the idea of linear approximation given by a standard derivative of functions of one variable to functions on Banach spaces.

Definition 5.8. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be Banach spaces, and* $\mathbf{U} \subset \mathbf{V}$ *be an open subset of* $\mathbf{V}$*. A function* $\mathbf{f} : \mathbf{U} \to \mathbf{W}$ *is called "Fréchet differentiable" at* $\boldsymbol{x} \in \mathbf{U}$ *if there exists a bounded linear operator* $\mathbf{f}'(\boldsymbol{x}) \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ *called a Fréchet derivative such that :*

$$\lim_{\|\boldsymbol{h}\| \to 0} \frac{\|\mathbf{f}(\boldsymbol{x} + \boldsymbol{h}) - \mathbf{f}(\boldsymbol{x}) - \mathbf{f}'(\boldsymbol{x})(\boldsymbol{h})\|}{\|\boldsymbol{h}\|} = 0.$$

We denote the Fréchet derivative by

$$\mathbf{f}'(\boldsymbol{x}) \equiv D_{\boldsymbol{x}}\mathbf{f}(\boldsymbol{x})$$

The Fréchet derivative in finite-dimensional spaces is basically the same as the Jacobi matrix. Higher derivatives are defined as follows:

$$D^n_{\boldsymbol{x}}\mathbf{f}(\boldsymbol{x}) \equiv D_{\boldsymbol{x}}(D^{n-1}_{\boldsymbol{x}})\mathbf{f}(\boldsymbol{x}) \quad \text{for} \quad n = 2, 3, ...$$

Since $D_{\boldsymbol{x}}\mathbf{f}(\boldsymbol{x}) \in \mathcal{L}(\mathbf{V}, \mathbf{W})$, so $D^2_{\boldsymbol{x}}\mathbf{f}(\boldsymbol{x}) \in \mathcal{L}(\mathbf{V}, \mathcal{L}(\mathbf{V}, \mathbf{W})) \equiv \mathcal{L}^2(\mathbf{V}, \mathbf{W})$. Actually, for any $\boldsymbol{P} \in \boldsymbol{V}$, we have $[D_{\boldsymbol{x}}\mathbf{f}(\boldsymbol{x})](\boldsymbol{P}) \in \boldsymbol{W}$,

The derivative of a composite function is calculated according to the chain rule:

$$D_{\boldsymbol{x}}(\mathbf{g} \circ \mathbf{f}) = D_{\boldsymbol{x}}\mathbf{g}(\mathbf{f}) \circ D_{\boldsymbol{x}}\mathbf{f} \equiv \mathbf{g}'(\mathbf{f}) \circ \mathbf{f}'$$

To show how to calculate the Fréchet derivative using the Definition 5.8, we present several examples. In these examples $\boldsymbol{E}$ and $\boldsymbol{P}$ denote 3×3 matrices.

Example 5.9. Let $f_1(\boldsymbol{E}) = tr\boldsymbol{E}$. For any $\boldsymbol{P}$ we have:

$$f_1(\boldsymbol{E} + \boldsymbol{P}) = tr(\boldsymbol{E} + \boldsymbol{P}) = tr\boldsymbol{E} + tr\boldsymbol{P}.$$

Hence

$$[D_{\boldsymbol{E}} f_1(\boldsymbol{E})](\boldsymbol{P}) = [D_{\boldsymbol{E}} tr(\boldsymbol{E})](\boldsymbol{P}) = tr\boldsymbol{P}.$$

Example 5.10. Let $\boldsymbol{f}_2(\boldsymbol{E}) = \boldsymbol{E}^2$. We have

$$(\boldsymbol{E}+\boldsymbol{P})^2 = \boldsymbol{E}^2 + \boldsymbol{E}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{E} + \boldsymbol{P}^2.$$

Hence

$$\left[D_{\boldsymbol{E}}\boldsymbol{f}_2(\boldsymbol{E})\right](\boldsymbol{P}) = \left[D_{\boldsymbol{E}}(\boldsymbol{E}^2)\right](\boldsymbol{P}) = \boldsymbol{E}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{E}.$$

Similarly we can calculate the second derivative

$$\begin{aligned}\left[D^2_{\boldsymbol{E}}\boldsymbol{f}_2(\boldsymbol{E})\right](\boldsymbol{P},\boldsymbol{P}) &= \left[D^2_{\boldsymbol{E}}(\boldsymbol{E}^2)\right](\boldsymbol{P},\boldsymbol{P}) = \left[D_{\boldsymbol{E}}(D_{\boldsymbol{E}}(\boldsymbol{E}^2))(\boldsymbol{P})\right](\boldsymbol{P})\\ &= \left[D_{\boldsymbol{E}}(\boldsymbol{E}\boldsymbol{P}+\boldsymbol{P}\boldsymbol{E})\right](\boldsymbol{P}) = \boldsymbol{P}^2 + \boldsymbol{P}^2 = 2\boldsymbol{P}^2\end{aligned}$$

or

$$\begin{aligned}\left[D^2_{\boldsymbol{E}}\boldsymbol{f}_2(\boldsymbol{E})\right](\boldsymbol{P}_1,\boldsymbol{P}_2) &= \left[D^2_{\boldsymbol{E}}(\boldsymbol{E}^2)\right](\boldsymbol{P}_1,\boldsymbol{P}_2) = \left[D_{\boldsymbol{E}}(D_{\boldsymbol{E}}(\boldsymbol{E}^2))(\boldsymbol{P}_1)\right](\boldsymbol{P}_2)\\ &= \left[D_{\boldsymbol{E}}(\boldsymbol{E}\boldsymbol{P}_1+\boldsymbol{P}_1\boldsymbol{E})\right](\boldsymbol{P}_2) = \boldsymbol{P}_2\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{P}_2.\end{aligned}$$

5.5 Reduction to Plane Waves

In this section we derive a general structure of plane waves elastodynamics for an arbitrary direction of propagation and for an arbitrary elastic solid regardless of its anisotropy. We follow the derivation developed in Domański and Young [64].

Given an arbitrary fixed unit vector $\boldsymbol{k}$, we consider motions of the type

$$\boldsymbol{x} = \boldsymbol{\pi}\boldsymbol{X} + \boldsymbol{u}(t,x), \tag{5.11}$$

where $\boldsymbol{\pi}$ is the gradient of a homogeneous background deformation (in particular $\boldsymbol{\pi}$ may be equal to $\boldsymbol{I}$), $\boldsymbol{u}$ is the superimposed displacement vector and

$$x = \boldsymbol{k}\cdot\boldsymbol{X}$$

is the projection of $\boldsymbol{X}$ on the direction $\boldsymbol{k}$. Since $\mathrm{Grad}\, x = \boldsymbol{k}$ the resulting deformation gradient is

$$\boldsymbol{F} = \boldsymbol{\pi} + \boldsymbol{d}\otimes\boldsymbol{k}, \tag{5.12}$$

where $\boldsymbol{d} = \boldsymbol{u}_{,x}$ is the vector displacement gradient.

Next, since the stress $\boldsymbol{S}$ now depends on x through $\boldsymbol{F}$ and $\boldsymbol{k}$ is independent of x, we calculate

$$\mathrm{Div}\,\boldsymbol{S} = (\boldsymbol{S}\,\boldsymbol{k})_{,x}. \tag{5.13}$$

Also,

$$\operatorname{Grad}\boldsymbol{v} = \boldsymbol{v}_{,x} \otimes \boldsymbol{k}, \quad \boldsymbol{F}_{,t} = \boldsymbol{d}_{,t} \otimes \boldsymbol{k}. \tag{5.14}$$

We now define the (reduced) stress vector $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{d})$ by

$$\boldsymbol{s}(\boldsymbol{d}) \equiv \boldsymbol{S}(\boldsymbol{F})\,\boldsymbol{k} = \boldsymbol{S}(\boldsymbol{\pi} + \boldsymbol{d} \otimes \boldsymbol{k})\,\boldsymbol{k}. \tag{5.15}$$

Then, on use of (5.13)–(5.15) in the equations of motion, we obtain the plane wave system

$$\rho_0 \boldsymbol{v}_{,t} = [\boldsymbol{s}(\boldsymbol{d})]_{,x}, \quad \boldsymbol{d}_{,t} = \boldsymbol{v}_{,x}, \tag{5.16}$$

which is a 6×6 system in one space variable x and time t. Note that we omitted the explicit dependence of $\boldsymbol{s}$ on $\boldsymbol{k}$.

In parallel with the definition of the stress vector $\boldsymbol{s}$ we define the (reduced) energy $V = V(\boldsymbol{d})$ via

$$V(\boldsymbol{d}) \equiv W(\boldsymbol{F}) = W(\boldsymbol{\pi} + \boldsymbol{d} \otimes \boldsymbol{k}), \tag{5.17}$$

from which we calculate

$$\boldsymbol{s}(\boldsymbol{d}) = D_{\boldsymbol{d}} V, \tag{5.18}$$

where $D_{\boldsymbol{d}} = \partial/\partial \boldsymbol{d}$, so that (5.16) may be rewritten as

$$\rho_0 \boldsymbol{v}_{,t} = (D_{\boldsymbol{d}} V)_{,x}, \quad \boldsymbol{d}_{,t} = \boldsymbol{v}_{,x}. \tag{5.19}$$

5.5.1 One–dimensional Quasi–linear System

We now restrict attention to the plane wave system (5.19), which, provided $V \in \mathrm{C}^2$, may be written in the quasi-linear form

$$\boldsymbol{w}_{,t} + \boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k})\boldsymbol{w}_{,x} = \boldsymbol{0}, \tag{5.20}$$

where

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{v}(t,x) \\ \boldsymbol{d}(t,x) \end{bmatrix}, \quad \boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k}) = -\begin{pmatrix} \boldsymbol{0} & \boldsymbol{B}(\boldsymbol{d}, \boldsymbol{k}) \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix} \tag{5.21}$$

and, for convenience, we have normalized the density by setting $\rho_0 = 1$. Here $\boldsymbol{B}$ is the symmetric matrix

$$\boldsymbol{B}(\boldsymbol{d}, \boldsymbol{k}) \equiv D_{\boldsymbol{d}} \boldsymbol{s} = D^2_{\boldsymbol{d}} V, \tag{5.22}$$

with components

$$B_{ij} = \frac{\partial^2 V(\boldsymbol{d})}{\partial d_i \partial d_j}.$$

In fact, it is straightforward to show that $\boldsymbol{B}$ is precisely the acoustic tensor:

$$\boldsymbol{B} = \boldsymbol{Q}(\boldsymbol{k}), \quad B_{ij} = Q_{ij}(\boldsymbol{k}) = \mathcal{A}_{ipjq} k_p k_q.$$

Thus, the assumption of strong ellipticity implies that the matrix $\boldsymbol{B}(\boldsymbol{d}, \boldsymbol{k})$ is positive definite. In particular, since $\boldsymbol{B}(\boldsymbol{d}, \boldsymbol{k})$ is symmetric and positive definite, $\boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k})$ always has a full set of eigenvectors and real eigenvalues. The system (5.20) is therefore hyperbolic. We will often suppress the dependence of these quantities on the vector $\boldsymbol{k}$, which is a fixed but arbitrary unit vector.

5.5.2 Eigensystems for Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

In order to find connections between the eigenvalues and eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$, we first prove a general lemma:

Lemma 5.11. *Let $\boldsymbol{A}$ be a $2m \times 2m$ matrix of the form*

$$\boldsymbol{A} = -\begin{pmatrix} \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{I}_m & \boldsymbol{0} \end{pmatrix}, \tag{5.23}$$

where $\boldsymbol{B}$ is a symmetric $m \times m$ matrix and $\boldsymbol{I}_m$ the identity $m \times m$ matrix. Then, for $i = 1, ..., m$, the eigensystem $\{\kappa_i, \boldsymbol{q}_i\}$ of the matrix $\boldsymbol{B}$ is related to the eigensystem $\{\lambda_j, \boldsymbol{r}_j\}$ of the matrix $\boldsymbol{A}$ through the connections

$$\lambda_{2i-1} = -\sqrt{\kappa_i} = -\lambda_{2i}, \tag{5.24}$$

$$\boldsymbol{r}_{2i-1} = \begin{bmatrix} \sqrt{\kappa_i}\boldsymbol{q}_i \\ \boldsymbol{q}_i \end{bmatrix}, \quad \boldsymbol{r}_{2i} = \begin{bmatrix} -\sqrt{\kappa_i}\boldsymbol{q}_i \\ \boldsymbol{q}_i \end{bmatrix}. \tag{5.25}$$

Proof. We have

$$\det(\lambda \boldsymbol{I}_{2m} - \boldsymbol{A}) = \det \begin{pmatrix} \lambda \boldsymbol{I}_m & \boldsymbol{B} \\ \boldsymbol{I}_m & \lambda \boldsymbol{I}_m \end{pmatrix} = \det(\lambda^2 \boldsymbol{I}_m - \boldsymbol{B}).$$

Thus, λ is an eigenvalue of the $\boldsymbol{A}$ if and only if $\kappa = \lambda^2$ is an eigenvalue of $\boldsymbol{B}$. Hence, in order to find the eigenvalues λ_j of the $2m \times 2m$ matrix $\boldsymbol{A}$ it suffices to find the eigenvalues κ_i of the $m \times m$ matrix $\boldsymbol{B}$ and then to obtain the eigenvalues of the matrix $\boldsymbol{A}$ through (5.24) for $i = 1, \ldots, m$.

Similarly for the corresponding eigenvectors. Assuming that $\boldsymbol{r}$ is the eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue λ, we have

$$(\lambda \boldsymbol{I}_{2m} - \boldsymbol{A})\boldsymbol{r} = \boldsymbol{0}.$$

We separate the $2m$-component eigenvector $\boldsymbol{r}$ into two m-component vectors $\boldsymbol{p}$ and $\boldsymbol{q}$:

$$\boldsymbol{r} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix}.$$

This gives

$$(\lambda \boldsymbol{I}_{2m} - \boldsymbol{A})\boldsymbol{r} = \begin{pmatrix} \lambda \boldsymbol{I}_m & \boldsymbol{B} \\ \boldsymbol{I}_m & \lambda \boldsymbol{I}_m \end{pmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = \boldsymbol{0},$$

and hence $\boldsymbol{p} = -\lambda \boldsymbol{q}$ and $\boldsymbol{q}$ is the eigenvector of the matrix $\boldsymbol{B}$ corresponding to the eigenvalue λ^2. Thus, in order to find an eigenvector $\boldsymbol{r}_j$ of the $(2m \times 2m)$ matrix $\boldsymbol{A}$ it is only necessary to find the corresponding eigenvector $\boldsymbol{q}_i$ of the smaller $(m \times m)$ matrix $\boldsymbol{B}$ and then to use the (5.25) with (5.24). Thanks to Lemma 5.11, we can reduce the calculation of the eigensystem for plane elastodynamic waves from a 6×6 system to a 3×3 system.

Remark 5.12. Denoting by $[\![\ ]\!]$ the largest integer that is less than or equal to a given number, we can express the correspondence between the eigensystem $\{\kappa_i, \boldsymbol{q}_i\}$ of $\boldsymbol{B}$ and the eigensystem $\{\lambda_j, \boldsymbol{r}_j\}$ of $\boldsymbol{A}$ in a more compact form. First let us notice that $i = [\![(j+1)/2]\!]$. With this notation we then have

$$\lambda_j = (-1)^j \sqrt{\kappa_i}, \quad \boldsymbol{r}_j = \begin{bmatrix} (-1)^j \sqrt{\kappa_i}\, \boldsymbol{q}_i \\ \boldsymbol{q}_i \end{bmatrix}. \tag{5.26}$$

5.5.3 Matrix $\boldsymbol{B}$ in terms of the strain energy

The structure of both the full system of elastodynamics, as well as of the plane wave system (5.19), is fully determined by the strain energy function $W(\boldsymbol{E})$ (and its derivatives). We recall that

$$\boldsymbol{S}(\boldsymbol{F}) = D_{\boldsymbol{F}} W,$$

so that the system is linear if W is quadratic as a function of $\boldsymbol{F}$. We can now describe the matrix $\boldsymbol{B}$ in terms of the strain energy $W(\boldsymbol{E})$, as follows. Please note that the reduced energy $V(\boldsymbol{d})$ can be expressed in terms of $W(\boldsymbol{E})$, where $\boldsymbol{E}$ is given by (5.2), and $\boldsymbol{F}$ by (5.12). Now from (5.22) using the chain and Leibniz rules, we get

$$\begin{aligned}\boldsymbol{B} &= D_{\boldsymbol{d}}\,(D_{\boldsymbol{d}} V) = D_{\boldsymbol{d}}\,(D_{\boldsymbol{E}} W\ D_{\boldsymbol{F}}\,\boldsymbol{E}\ D_{\boldsymbol{d}}\,\boldsymbol{F}) \\ &= D^2_{\boldsymbol{E}} W \circ D_{\boldsymbol{F}}\,\boldsymbol{E}\ D_{\boldsymbol{d}}\,\boldsymbol{F} + D_{\boldsymbol{E}} W\ D^2_{\boldsymbol{F}}\,\boldsymbol{E} \circ D_{\boldsymbol{d}}\,\boldsymbol{F},\end{aligned} \tag{5.27}$$

since $\boldsymbol{F}(\boldsymbol{d})$ is linear. Here $\circ$ means composition in each component, thus for any $\boldsymbol{L}$

$$D^2_{\boldsymbol{E}} W \circ \boldsymbol{L}(\boldsymbol{b}_1, \boldsymbol{b}_2) = D^2_{\boldsymbol{E}} W(\boldsymbol{L}\boldsymbol{b}_1, \boldsymbol{L}\boldsymbol{b}_2). \tag{5.28}$$

Next we calculate the individual derivatives from (5.27): for any fixed vector $\boldsymbol{q}$

$$(D_{\boldsymbol{d}}\,\boldsymbol{F})(\boldsymbol{q}) = \boldsymbol{q} \otimes \boldsymbol{k} \quad \text{and} \quad D^2_{\boldsymbol{d}}\,\boldsymbol{F} = \boldsymbol{0}, \tag{5.29}$$

while for any matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$

$$(D_{\boldsymbol{F}}\,\boldsymbol{E})(\boldsymbol{P}) = \frac{1}{2}\,(\boldsymbol{F}^t\,\boldsymbol{P} + \boldsymbol{P}^t\,\boldsymbol{F}) \quad \text{and} \tag{5.30}$$

$$(D^2_{\boldsymbol{F}}\,\boldsymbol{E})(\boldsymbol{P}, \boldsymbol{Q}) = \frac{1}{2}\,(\boldsymbol{P}^t\,\boldsymbol{Q} + \boldsymbol{Q}^t\,\boldsymbol{P}). \tag{5.31}$$

We thus calculate

$$D_{\boldsymbol{F}}\,\boldsymbol{E}\ D_{\boldsymbol{d}}\,\boldsymbol{F}(\boldsymbol{q}) \equiv \boldsymbol{\mathcal{P}}_{\boldsymbol{q}}. \tag{5.32}$$

From (5.29), (5.30), and (5.32), after some algebra we get

$$\boldsymbol{\mathcal{P}}_{\boldsymbol{q}} = \frac{1}{2}\,(\boldsymbol{\pi}^t\,\boldsymbol{q} \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \boldsymbol{q}\,\boldsymbol{\pi}) + (\boldsymbol{d} \cdot \boldsymbol{q})\ \boldsymbol{k} \otimes \boldsymbol{k}. \tag{5.33}$$

Similarly from (5.31) and (5.29)

$$(D^2_{\boldsymbol{F}}\,\boldsymbol{E} \circ D_{\boldsymbol{d}}\,\boldsymbol{F})(\boldsymbol{q}_1, \boldsymbol{q}_2) = (\boldsymbol{q}_1 \cdot \boldsymbol{q}_2)\ \boldsymbol{k} \otimes \boldsymbol{k}, \tag{5.34}$$

and substituting these into (5.27), we get

$$\boldsymbol{q}_1 \cdot \boldsymbol{B} \cdot \boldsymbol{q}_2 = D^2_{\boldsymbol{E}} W(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_1}, \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_2}) + (\boldsymbol{q}_1 \cdot \boldsymbol{q}_2)\ D_{\boldsymbol{E}} W(\boldsymbol{k} \otimes \boldsymbol{k}). \tag{5.35}$$

Thus once the strain energy function $W(\boldsymbol{E})$ is specified, the matrix $\boldsymbol{B}$ can be directly found from (5.35) by differentiating. We have derived the general structure of plane waves elastodynamics for arbitrary direction $\boldsymbol{k}$ and arbitrary anisotropic material.

6

Nonlinear Isotropic Material

To solve the system (5.10) it is necessary to specify the strain-energy function. Here, we consider an isotropic material, for which the energy is a function of three independent deformation or strain invariants, which we denote by (I_1, I_2, I_3). We define these as the principal invariants of the Green strain tensor

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}} + \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}). \tag{6.1}$$

Thus,

$$\begin{aligned} I_1 \equiv I_1(\boldsymbol{E}) \equiv I_{\mathbf{E}} &= tr\boldsymbol{E}, \\ I_2 \equiv I_2(\boldsymbol{E}) \equiv II_{\mathbf{E}} &= \tfrac{1}{2}\left[(tr\boldsymbol{E})^2 - tr(\boldsymbol{E})^2\right], \\ I_3 \equiv I_3(\boldsymbol{E}) \equiv III_{\mathbf{E}} &= det\boldsymbol{E}. \end{aligned} \tag{6.2}$$

The other commonly used invariants are those of the Cauchy - Green deformation tensor $\boldsymbol{C}$:

$$\begin{aligned} I_1(\boldsymbol{C}) \equiv I_{\mathbf{C}} &= tr\boldsymbol{C}, \\ I_2(\boldsymbol{C}) \equiv II_{\mathbf{C}} &= \tfrac{1}{2}\left[(tr\boldsymbol{C})^2 - tr(\boldsymbol{C})^2\right], \\ I_3(\boldsymbol{C}) \equiv III_{\mathbf{C}} &= det\boldsymbol{C}. \end{aligned} \tag{6.3}$$

It should be emphasized that the notation I_1, I_2, I_3 is frequently used for the principal invariants of $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$ rather than $\boldsymbol{E}$, but here it is convenient to adopt the notation (6.2).

On the other hand the isotropy assumption implies that W depends on $\boldsymbol{F}$ only through the stretch tensor, say the right Cauchy - Green tensor $\boldsymbol{U}$. This can be seen from the objectivity assumption and the polar decomposition (5.1), since we have:

$$W(\boldsymbol{F}) = W(\boldsymbol{R}^T\boldsymbol{F}) = W(\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U}) = W(\boldsymbol{U}).$$

Hence in the isotropic material the energy function may be represented as a symmetric function of three stretches – the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the stretch tensor $\boldsymbol{U}$:

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_3, \lambda_1) = W(\lambda_3, \lambda_1, \lambda_2).$$

Equivalently W can be written as a function of three principal invariants i_1, i_2, i_3 of the stretch tensor $\boldsymbol{U}$:

$$W = W(i_1, i_2, i_3),$$

where the invariants i_1, i_2, i_3 are related to stretches in the following way:

$$\begin{aligned} i_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ i_2 &= \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1, \\ i_3 &= \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{6.4}$$

On the other hand, the principal invariants of the tensor $\boldsymbol{C} = \boldsymbol{U}^2$ are expressed in terms of stretches as follows

$$\begin{aligned} I_1(\boldsymbol{C}) &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2(\boldsymbol{C}) &= \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \\ I_3(\boldsymbol{C}) &= \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{6.5}$$

Moreover, we have

$$\begin{aligned} I_1(\boldsymbol{C}) &= i_1^2 - 2i_2, \\ I_2(\boldsymbol{C}) &= i_2^2 - 2i_1 i_2, \\ I_3(\boldsymbol{C}) &= i_3^2. \end{aligned} \tag{6.6}$$

6.1 Murnaghan Material

The expression for the strain energy function for an isotropic solid in terms of strain invariants including the third order terms was given by Murnaghan [126] in the following form:

$$W = \frac{1}{2}(\lambda_L + 2\mu_L){I_{\mathbf{E}}}^2 - 2\mu_L II_{\mathbf{E}} + \\ \frac{1}{3}(l_M + 2m_M){I_{\mathbf{E}}}^3 - 2m_M I_{\mathbf{E}} II_{\mathbf{E}} + n_M III_{\mathbf{E}}. \tag{6.7}$$

In (6.7) λ_L, μ_L are second order Lame's constants, and l_M, m_M, n_M are third order Murnaghan's constants. Using the explicit form of the invariants (6.2) in terms of the strain Cartesian coordinates:

$$\begin{aligned} I_{\mathbf{E}} &= E_{11} + E_{22} + E_{33}, \\ II_{\mathbf{E}} &= det \begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{bmatrix} + det \begin{bmatrix} E_{33} & E_{31} \\ E_{13} & E_{11} \end{bmatrix} + det \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \\ III_{\mathbf{E}} &= det \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \end{aligned} , \tag{6.8}$$

we can write the Murnaghan form of the energy as follows:

$$\begin{aligned} W = & \frac{\lambda_L}{2}\left(E_{11} + E_{22} + E_{33}\right)^2 + \mu_L\left(E_{11}^2 + E_{22}^2 + E_{33}^2 + 2E_{12}^2 + 2E_{13}^2 + 2E_{23}^2\right) \\ & + \frac{l_M}{3}\left[E_{11}\left({E_{11}}^2 + {E_{12}}^2 + {E_{13}}^2\right) + 2\,E_{12}\left(E_{11}\,E_{12} + E_{12}\,E_{22} + E_{13}\,E_{23}\right)\right. \\ & + E_{22}\left({E_{12}}^2 + {E_{22}}^2 + {E_{23}}^2\right) + 2\,E_{13}\left(E_{11}\,E_{13} + E_{12}\,E_{23} + E_{13}\,E_{33}\right) \\ & \left. + E_{33}\left({E_{13}}^2 + {E_{23}}^2 + {E_{33}}^2\right) + 2\,E_{23}\left(E_{12}\,E_{13} + E_{22}\,E_{23} + E_{23}\,E_{33}\right)\right] \\ & + m_M\left(E_{11} + E_{22} + E_{33}\right)\left[\frac{1}{3}\left(E_{11} + E_{22} + E_{33}\right)^2\right. \\ & \left. - 2\left({E_{12}}^2 + {E_{13}}^2 + {E_{23}}^2\right) - 2\left(E_{11}\,E_{22} + E_{11}\,E_{33} + E_{22}\,E_{33}\right)\right] \\ & + n_M\left(E_{11}\,E_{22}\,E_{33} + 2\,E_{12}\,E_{13}\,E_{23} - E_{11}\,{E_{23}}^2 - E_{22}\,{E_{13}}^2 - E_{33}\,{E_{12}}^2\right). \end{aligned}$$

Remark 6.1. The other commonly used expression for the strain energy with third order terms is due to L. D. Landau [107]:

$$W = \frac{1}{2}\lambda_L J_1^2 + \mu_L J_2 + \frac{A}{3}J_3 + BJ_1J_2 + \frac{C}{3}J_1^3$$

where J_1, J_2 and J_3 are strain invariants given by the formulas: $J_1 = tr\mathbf{E}$, $J_2 = tr\mathbf{E}^2$, $J_3 = tr\mathbf{E}^3$, and A, B, C, are third order constants expressed in terms of Murnaghan's constants as follows: $A = n_M, B = m_M - \dfrac{n_M}{2}, C = l_M - m_M + \dfrac{n_M}{2}$.

6.2 St. Venant–Kirchhoff Material

If we restrict ourselves to only linear terms in (6.7), we have an isotropic physically (materially) linear theory (see, e.g., Ciarlet [32], p. 155):

$$W = \frac{1}{2}(\lambda_L + 2\mu_L) I_{\mathbf{E}}^{\,2} - 2\mu_L II_{\mathbf{E}}.$$

The medium with such equations of state is called a Saint–Venant Kirchhoff material. In this subsection we will prove that the energy function of the isotropic Saint–Venant Kirchhoff material is *not polyconvex* ([32]). The proof will follow from the Lemma below. Let us first notice that this energy can be represented as

$$\begin{aligned} W(\boldsymbol{F}) = W(\boldsymbol{E}) = \frac{\lambda_L}{2}\,(tr\boldsymbol{E})^2 + \mu_L\, tr(\boldsymbol{E}^2) = \\ \frac{\lambda_L}{2}\left[\frac{(tr\boldsymbol{C})^2}{4} - \frac{3tr\boldsymbol{C}}{2} + \frac{9}{4}\right] + \mu_L \left[\frac{(tr\boldsymbol{C})^2}{4} - \frac{tr\boldsymbol{C}}{2} + \frac{3}{4}\right] = \\ -\frac{3\lambda_L + 2\mu_L}{4}\, tr\boldsymbol{C} + \frac{\lambda_L + 2\mu_L}{8}\, tr\boldsymbol{C}^{\,2} + \frac{\lambda_L}{4}\, tr(cof\boldsymbol{C}) + \frac{9\lambda_L + 6\mu_L}{8}\,. \end{aligned}$$

Lemma 6.2. *The energy function of the form*

$$W(\boldsymbol{F}) = \alpha\, tr\boldsymbol{C} + \beta\, tr\boldsymbol{C}^{\,2} + \gamma\, tr(cof\boldsymbol{C}) \qquad (6.9)$$

where $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, *and* $\beta > 0, \gamma > 0$, *is* **not polyconvex** *if* $\alpha < 0$.

Proof:

For any $\epsilon > 0$, let us take two matrices

$$\boldsymbol{F}_1^\epsilon = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad \text{and} \quad \boldsymbol{F}_2^\epsilon = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 3\epsilon \end{pmatrix}.$$

We will show that for $\alpha < 0$ the following inequality does not hold:

$$W\left(\frac{\boldsymbol{F}_1^\epsilon + \boldsymbol{F}_2^\epsilon}{2}\right) \leq \frac{1}{2}\left(W\left(\boldsymbol{F}_1^\epsilon\right) + W\left(\boldsymbol{F}_2^\epsilon\right)\right). \tag{6.10}$$

We have

$$W\left(\frac{\boldsymbol{F}_1^\epsilon + \boldsymbol{F}_2^\epsilon}{2}\right) = \alpha\, tr\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 4\epsilon^2 \end{pmatrix} + \beta tr\begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & 16\epsilon^4 \end{pmatrix}$$

$$+ \gamma tr\begin{pmatrix} 4\epsilon^4 & 0 & 0 \\ 0 & 4\epsilon^4 & 0 \\ 0 & 0 & \epsilon^4 \end{pmatrix}$$

$$= 6\,\alpha\,\epsilon^2 + 18\,\beta\,\epsilon^4 + 9\,\gamma\,\epsilon^4.$$

On the other hand

$$W\left(\boldsymbol{F}_1^\epsilon\right) = \alpha\, tr\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix} + \beta tr\begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon^4 \end{pmatrix} + \gamma tr\begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon^4 \end{pmatrix}$$

$$= 3\,(\alpha\,\epsilon^2 + \,\beta\,\epsilon^4 + \,\gamma\,\epsilon^4),$$

and

$$W\left(\boldsymbol{F}_2^\epsilon\right) = \alpha\, tr\begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^2 & 0 \\ 0 & 0 & 9\epsilon^2 \end{pmatrix} + \beta tr\begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & 81\epsilon^4 \end{pmatrix} + \gamma tr\begin{pmatrix} 9\epsilon^4 & 0 & 0 \\ 0 & 9\epsilon^4 & 0 \\ 0 & 0 & \epsilon^4 \end{pmatrix}$$

$$= 11\,\alpha\,\epsilon^2 + 83\,\beta\,\epsilon^4 + \,19\gamma\,\epsilon^4.$$

Hence (6.10) is equivalent to the inequality

$$6\,\alpha\,\epsilon^2 + 18\,\beta\,\epsilon^4 + 9\,\gamma\,\epsilon^4 \le 7\,\alpha\,\epsilon^2 + 43\,\beta\,\epsilon^4 + 11\,\gamma\,\epsilon^4$$

which in turn is equivalent to

$$0 \le \alpha + (25\,\beta + 2\,\gamma)\epsilon^2 \equiv \varphi(\epsilon).$$

However, the last inequality does not hold if $\alpha < 0$, and ϵ is sufficiently small (see Fig. 5.1).

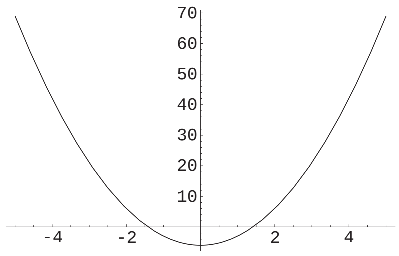

Fig. 6.1. The schematic graph of the function $\varphi(\epsilon)$ with $\alpha < 0$.

Since (6.10) does not hold, hence $W(\boldsymbol{F})$ from (6.9) is not polyconvex. Taking into account the form of the stored energy for the St. Venant–Kirchhoff material, we conclude that the energy function for the St. Venant–Kirchhoff material is not polyconvex.

Because of the nonlinear dependence of the energy on the gradient of displacement, the system (5.10), since it is quasi-linear and hyperbolic, does not in general have smooth solutions even for smooth initial or boundary data. Typically, shock waves form in a finite time. To avoid formation of singularities, Sideris [152] (see also Agemi [2]) introduced a *null condition*. Under this condition the following theorem was proved.

Theorem 6.3. *Assume that the* ***null condition*** *is satisfied. Then, there exists a positive constant* ε_0 *such that the initial value problem for the system (5.10)*

$$\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{U}} = \mathbf{0}, \qquad \boldsymbol{\mathcal{U}}|_{t=0} = \varepsilon\boldsymbol{\mathcal{U}}_0$$

with $\boldsymbol{\mathcal{U}}_0 \in \mathrm{C}_0^\infty(\mathbb{R}^3)$, has a unique global-in-time C^∞ solution $\boldsymbol{\mathcal{U}}$ for any $\varepsilon \le \varepsilon_0$.

In the next chapter we will present the derivation of the null condition and propose some new alternative forms of the null condition.

7

Null Condition

7.1 Derivation of the Null Condition

We now briefly recall the ideas underlying the derivation of the null condition in the form provided by Sideris [152] and Agemi [2]. This form of the null condition is expressed in terms of the derivatives of the strain-energy function with respect to the strain invariants. First, we write the equation of motion (5.3) as a second-order system with displacement as the dependent variable. This gives

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mathrm{Div}\left(\frac{\partial W}{\partial \boldsymbol{D}}\right), \tag{7.1}$$

and in component form this can be expanded as

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \mathcal{A}_{ijkl} D_{kl,j} \equiv \mathcal{A}_{ijkl} u_{k,lj}, \tag{7.2}$$

where the coefficients $\mathcal{A}_{ijkl}$ are as defined in (5.6), but now written as

$$\mathcal{A}_{ijkl} = \frac{\partial^2 W}{\partial D_{ij} \partial D_{kl}}. \tag{7.3}$$

It suffices to consider explicitly only terms of first and second order in the components of $\boldsymbol{D}$ and its derivatives, and we therefore expand $\mathcal{A}_{ijkl}$ to the first order in the form

$$\mathcal{A}_{ijkl} = \mathcal{A}^0_{ijkl} + \mathcal{B}^0_{ijklmn} D_{mn}, \tag{7.4}$$

where the superscript ^{0} signifies evaluation in the reference configuration $\boldsymbol{D} = \boldsymbol{O}$ and

$$\mathcal{B}_{ijklmn} = \frac{\partial^3 W}{\partial D_{ij} \partial D_{kl} \partial D_{mn}} \tag{7.5}$$

are the components of the tensor $\boldsymbol{\mathcal{B}}$. To the considered order, equation (7.2) then takes the form

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \mathcal{A}^0_{ijkl} u_{k,jl} = \mathcal{B}^0_{ijklmn} D_{mn} D_{kl,j}, \tag{7.6}$$

wherein the linear terms are on the left-hand side and the nonlinear terms (truncated at second order) are on the right-hand side.

Now, since we are considering an isotropic material, we write $W = W(I_1, I_2, I_3)$ and it follows, using the component form of (5.8) and the formula (7.3), that

$$S_{ij} = \frac{\partial W}{\partial D_{ij}} = \sum_{p=1}^{3} W_p \frac{\partial I_p}{\partial D_{ij}}, \tag{7.7}$$

$$\mathcal{A}_{ijkl} = \sum_{p,q=1}^{3} W_{pq} \frac{\partial I_p}{\partial D_{ij}} \frac{\partial I_q}{\partial D_{kl}} + \sum_{p=1}^{3} W_p \frac{\partial^2 I_p}{\partial D_{ij} \partial D_{kl}}, \tag{7.8}$$

where $W_p = \partial W / \partial I_p, W_{pq} = \partial^2 W / \partial I_p \partial I_q$, and similarly, on use of (7.5),

$$\begin{aligned} \mathcal{B}_{ijklmn} = & \sum_{p,q,r=1}^{3} W_{pqr} \frac{\partial I_p}{\partial D_{ij}} \frac{\partial I_q}{\partial D_{kl}} \frac{\partial I_r}{\partial D_{mn}} \\ & + \sum_{p,q=1}^{3} W_{pq} \left(\frac{\partial I_p}{\partial D_{ij}} \frac{\partial^2 I_q}{\partial D_{kl} \partial D_{mn}} + \frac{\partial I_p}{\partial D_{kl}} \frac{\partial^2 I_q}{\partial D_{ij} \partial D_{mn}} \right. \\ & \left. + \frac{\partial I_p}{\partial D_{mn}} \frac{\partial^2 I_q}{\partial D_{ij} \partial D_{kl}} \right) + \sum_{p=1}^{3} W_p \frac{\partial^3 I_p}{\partial D_{ij} \partial D_{kl} \partial D_{mn}}, \end{aligned} \tag{7.9}$$

with $W_{pqr} = \partial^3 W / \partial I_p \partial I_q \partial I_r$.

We require the values of these expressions in the reference configuration, for which purpose we need to calculate the derivatives of I_p, $p = 1, 2, 3$, with respect to the components of $\boldsymbol{D}$. We first note that the invariants can be expanded in terms of $\boldsymbol{D}$ in the forms

$$I_1 = \mathrm{tr}\,\boldsymbol{D} + \frac{1}{2}\mathrm{tr}\,(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}), \tag{7.10}$$

$$\begin{aligned} I_2 = &\ \frac{1}{4}\left[2(\mathrm{tr}\,\boldsymbol{D})^2 - \mathrm{tr}\,(\boldsymbol{D}^2) - \mathrm{tr}\,(\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}})\right] \\ &+ \frac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{D})\,\mathrm{tr}\,(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}) - \mathrm{tr}\,(\boldsymbol{D}^2\boldsymbol{D}^{\mathrm{T}})\right], \end{aligned} \tag{7.11}$$

$$\begin{aligned} I_3 = &\ \frac{1}{12}\left[2(\mathrm{tr}\,\boldsymbol{D})^3 + \mathrm{tr}\,(\boldsymbol{D}^3)\right] \\ &- \frac{1}{4}\left[(\mathrm{tr}\,\boldsymbol{D})\,\mathrm{tr}\,(\boldsymbol{D}^2) + (\mathrm{tr}\,\boldsymbol{D})\,\mathrm{tr}\,(\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}) - \mathrm{tr}\,(\boldsymbol{D}^2\boldsymbol{D}^{T})\right], \end{aligned} \tag{7.12}$$

where I_2 and I_3 have been truncated at the third order in $\boldsymbol{D}$. To the second order the first derivatives of these invariants are

$$\frac{\partial I_1}{\partial \boldsymbol{D}} = \boldsymbol{I} + \boldsymbol{D}, \tag{7.13}$$

$$\begin{aligned} \frac{\partial I_2}{\partial \boldsymbol{D}} = &\ (\mathrm{tr}\,\boldsymbol{D})\boldsymbol{I} - \frac{1}{2}(\boldsymbol{D} + \boldsymbol{D}^{\mathrm{T}}) + \frac{1}{2}\mathrm{tr}\,(\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}})\boldsymbol{I} + (\mathrm{tr}\,\boldsymbol{D})\boldsymbol{D} \\ &- \frac{1}{2}\left(\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} + \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} + \boldsymbol{D}^2\right), \end{aligned} \tag{7.14}$$

$$\begin{aligned} \frac{\partial I_3}{\partial \boldsymbol{D}} = &\ \frac{1}{2}(\mathrm{tr}\,\boldsymbol{D})\left[(\mathrm{tr}\,\boldsymbol{D})\boldsymbol{I} - \boldsymbol{D} - \boldsymbol{D}^{\mathrm{T}}\right] \\ &+ \frac{1}{4}\left(\boldsymbol{D}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}} + \boldsymbol{D}\boldsymbol{D}^{\mathrm{T}} + \boldsymbol{D}^{\mathrm{T}}\boldsymbol{D} + \boldsymbol{D}^2\right) \\ &- \frac{1}{4}\left[\mathrm{tr}\,(\boldsymbol{D}^2) + \mathrm{tr}\,(\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}})\right]\boldsymbol{I}. \end{aligned} \tag{7.15}$$

Corresponding second- and third-order derivatives may also be written down but they are mostly quite lengthy so are not listed here. More specifically, in the reference configuration the first derivatives reduce simply to

$$\frac{\partial I_1}{\partial \boldsymbol{D}} = \boldsymbol{I}, \quad \frac{\partial I_2}{\partial \boldsymbol{D}} = \frac{\partial I_3}{\partial \boldsymbol{D}} = \boldsymbol{O}, \tag{7.16}$$

and, in component form, the second derivatives are

$$\frac{\partial^2 I_1}{\partial D_{ij}\partial D_{kl}} = \delta_{ik}\delta_{jl}, \quad \frac{\partial^2 I_2}{\partial D_{ij}\partial D_{kl}} = \delta_{ij}\delta_{kl} - \frac{1}{2}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}), \quad \frac{\partial^2 I_3}{\partial D_{ij}\partial D_{kl}} = 0, \tag{7.17}$$

where δ_{ij} is the Kronecker delta, while the third derivative of I_1 vanishes and, in the reference configuration, the third derivatives of I_2 and I_3 are given by

$$\frac{\partial^3 I_2}{\partial D_{ij}\partial D_{kl}\partial D_{mn}} = \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{kl}\delta_{im}\delta_{jn} - \frac{1}{2}(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{lm}\delta_{jn} + \delta_{jl}\delta_{im}\delta_{kn} + \delta_{jl}\delta_{km}\delta_{in} + \delta_{il}\delta_{km}\delta_{jn} + \delta_{im}\delta_{jk}\delta_{ln}), \tag{7.18}$$

$$\frac{\partial^3 I_3}{\partial D_{ij}\partial D_{kl}\partial D_{mn}} = \delta_{ij}\delta_{kl}\delta_{mn} - \frac{1}{2}(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{lm}\delta_{kn} + \delta_{kl}\delta_{im}\delta_{jn} + \delta_{kl}\delta_{jm}\delta_{in} + \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{jk}\delta_{il}\delta_{mn}) + \frac{1}{4}(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{jk}\delta_{lm}\delta_{in} + \delta_{jk}\delta_{im}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{jl}\delta_{km}\delta_{in} + \delta_{il}\delta_{km}\delta_{jn} + \delta_{jl}\delta_{im}\delta_{kn} + \delta_{ik}\delta_{lm}\delta_{jn}). \tag{7.19}$$

In the reference configuration we assume that both $W = W^0$ and $\boldsymbol{S} = \boldsymbol{S}^0$ vanish, and since, on use of the above results in (7.7), $\boldsymbol{S}^0 = W_1^0\boldsymbol{I}$, we have

$$W^0 = 0, \quad W_1^0 = 0, \tag{7.20}$$

where again and in what follows the superscript ^{0} indicates evaluation in the reference configuration. Also, from (7.8), we obtain

$$\mathcal{A}^0_{ijkl} = \left(W^0_{11} + W^0_2\right)\delta_{ij}\delta_{kl} - \frac{1}{2}W^0_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

The corresponding expression for $\mathcal{B}^0_{ijklmn}$ may be obtained from (7.9) and (7.16)–(7.19) but is not given explicitly here.

The nonlinear terms are quite complicated and we do not list them all here since most are not relevant to the argument. Let us now set $\rho_0 c_l^2 \equiv W^0_{11}$ and $\rho_0 c_s^2 \equiv -\frac{1}{2}W^0_2$, where c_l and c_s are the speeds of the linearized longitudinal and shear waves, respectively. Noting that $\boldsymbol{D} = \mathrm{Grad}\,\boldsymbol{u}$, the equations of motion (7.1) can be written as

$$\boldsymbol{u}_{tt} - c_s^2\triangle\boldsymbol{u} - (c_l^2 - c_s^2)\nabla(\nabla\cdot\boldsymbol{u}) = N(\nabla\boldsymbol{u}, \nabla^2\boldsymbol{u}), \tag{7.21}$$

where N is a nonlinear function of its arguments and the operator ∇ has replaced Grad. On the left-hand side is the classical linear operator of elastodynamics, while on the right-hand side we have second-order and higher-order terms. Amongst the nonlinear terms only terms of the form $\nabla(\nabla\cdot\boldsymbol{u})^2$ interfere with the energy estimates needed in the proof of global existence of a smooth solution to elastodynamic equations (5.3) (see T. Sideris [152] and R. Agemi [2] for more details). We display this term explicitly and write

$$N(\nabla \boldsymbol{u}, \nabla^2 \boldsymbol{u}) = \frac{1}{2\rho_0}(3W_{11}^0 + W_{111}^0)\nabla(\nabla \cdot \boldsymbol{u})^2 + ...$$

The null condition is imposed to cancel this term.

Definition 7.1. *We say that the elastodynamic equations (5.3) or (7.1) satisfy the* ***null condition*** *if*

$$\left(3W_{11}^0 + W_{111}^0\right) \equiv \left(3\frac{\partial^2 W}{\partial I_1^2} + \frac{\partial^3 W}{\partial I_1^3}\right)\Bigg|_{I_1 = I_2 = I_3 = 0} = 0. \tag{7.22}$$

Remark 7.2. Actually, the condition (7.22) is slightly different than the one introduced by T. Sideris [152] and R. Agemi [2]. In fact, in their formulation of the null condition, they used invariants of a different strain tensor than we did. In particular, they took the invariant $K_1 = tr(2\boldsymbol{E})$, while we used the invariant $I_1 = tr\boldsymbol{E}$. It is easy to show that our condition (7.22) is equivalent to the condition

$$\left(3\frac{\partial^2 W}{\partial K_1^2} + 2\frac{\partial^3 W}{\partial K_1^3}\right)\Bigg|_{K_1 = K_2 = K_3 = 0} = 0. \tag{7.23}$$

formulated by T. Sideris [152] and R. Agemi [2]. In fact since $2I_1 = K_1$, hence we have

$$\begin{aligned}
\frac{\partial W}{\partial K_1} &= \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial K_1} = \frac{1}{2}\frac{\partial W}{\partial I_1} \\
\frac{\partial^2 W}{\partial K_1^2} &= \frac{\partial}{\partial K_1}\left(\frac{\partial W}{\partial K_1}\right) = \frac{1}{4}\frac{\partial^2 W}{\partial I_1^2} \\
\frac{\partial^3 W}{\partial K_1^3} &= \frac{\partial}{\partial K_1}\left(\frac{\partial^2 W}{\partial K_1^2}\right) = \frac{1}{8}\frac{\partial^3 W}{\partial I_1^3}.
\end{aligned}$$

Therefore

$$3\frac{\partial^2 W}{\partial K_1^2} + 2\frac{\partial^3 W}{\partial K_1^3} = \frac{1}{4}\left(3\frac{\partial^2 W}{\partial I_1^2} + \frac{\partial^3 W}{\partial I_1^3}\right).$$

Hence (7.23) is equivalent to (7.22).

7.2 Alternative Forms of the Null Condition

In this and the following sections we formulate alternative forms of the null condition. It turns out (see [62] and [170]) that the null condition can be expressed with the use of plane waves. The null condition asserts that each of the nonlinear elastic plane waves is *not* genuinely nonlinear in the reference configuration or, equivalently, that quadratically nonlinear self-interaction of plane waves is forbidden.

7.2.1 Null Condition in Terms of the Eigensystem of the Matrix $\boldsymbol{B}$

In order to formulate the null condition in a simpler form, we now prove the following lemma:

Lemma 7.3. *Let*

$$\boldsymbol{A}(\boldsymbol{w}) = -\begin{pmatrix} \mathbf{0} & \boldsymbol{B}(\boldsymbol{d}) \\ \boldsymbol{I} & \mathbf{0} \end{pmatrix} \tag{7.24}$$

be the matrix from the formula (12.6), with the argument $\boldsymbol{k}$ *omitted and with* $\boldsymbol{w} = [\boldsymbol{v}, \boldsymbol{d}]^{\mathrm{T}}$*. Suppose that* $\kappa_i(\boldsymbol{d}) \neq 0$ *are the eigenvalues of* $\boldsymbol{B}(\boldsymbol{d})$*,* $\boldsymbol{q}_i(\boldsymbol{d})$ *the corresponding eigenvectors* $(i = 1, 2, 3)$*, and that* $\lambda_j(\boldsymbol{w})$ *are the eigenvalues of* $\boldsymbol{A}(\boldsymbol{w})$ *with* $\boldsymbol{r}_j(\boldsymbol{w})$ *the corresponding eigenvectors* $(j = 1, \dots, 6)$*. Then*

$$(D_{\boldsymbol{w}}\, \lambda_j(\boldsymbol{w})) \cdot \boldsymbol{r}_j = 0 \iff (D_{\boldsymbol{d}}\, \kappa_i(\boldsymbol{d})) \cdot \boldsymbol{q}_i = 0. \tag{7.25}$$

Proof. Taking into account Lemma 5.11, we calculate

$$\begin{aligned}(D_{\boldsymbol{w}}\, \lambda_{2i-1}) \cdot \boldsymbol{r}_{2i-1} &= (D_{\boldsymbol{w}}\, (-\sqrt{\kappa_i})) \cdot \begin{bmatrix} \sqrt{\kappa_i}\boldsymbol{q}_i \\ \boldsymbol{q}_i \end{bmatrix} \\ &= \left[\mathbf{0}, -\frac{1}{2\sqrt{\kappa_i}}(D_{\boldsymbol{d}}\, \kappa_i)\right]^{\mathrm{T}} \cdot \begin{bmatrix} \sqrt{\kappa_i}\boldsymbol{q}_i \\ \boldsymbol{q}_i \end{bmatrix} = -\frac{1}{2\sqrt{\kappa_i}}(D_{\boldsymbol{d}}\, \kappa_i) \cdot \boldsymbol{q}_i. \end{aligned} \tag{7.26}$$

Similarly,

$$\begin{aligned}(D_{\boldsymbol{w}}\, \lambda_{2i}) \cdot \boldsymbol{r}_{2i} &= (D_{\boldsymbol{w}}\, \sqrt{\kappa_i}) \cdot \begin{bmatrix} -\sqrt{\kappa_i}\boldsymbol{q}_i \\ \boldsymbol{q}_i \end{bmatrix} \\ &= \left[\mathbf{0}, \frac{1}{2\sqrt{\kappa_i}}(D_{\boldsymbol{d}}\, \kappa_i)\right]^{\mathrm{T}} \cdot \begin{bmatrix} -\sqrt{\kappa_i}\boldsymbol{q}_i \\ \boldsymbol{q}_i \end{bmatrix} = \frac{1}{2\sqrt{\kappa_i}}(D_{\boldsymbol{d}}\, \kappa_i) \cdot \boldsymbol{q}_i. \end{aligned} \tag{7.27}$$

Hence (7.25).

We can now state that the *null condition for elastodynamics* can be formulated as

$$\left(D_{\boldsymbol{d}}\,\kappa_i \cdot \boldsymbol{q}_i\right)\Big|_{\boldsymbol{d}\,=\,\boldsymbol{0}} = 0. \quad \text{for } i = 1, 2, 3. \tag{7.28}$$

Application to the Murnaghan Material

Here we illustrate the condition (7.28) for the Murnaghan isotropic elastic material, for which the strain-energy function may be written as (6.7)(see Murnaghan [126])

We express the energy function (6.7) in terms of the displacement gradient, truncating the resulting expression so that the stress tensor has only quadratically nonlinear terms. We then restrict attention to plane waves propagating in the $[1, 0, 0]$ direction. We obtain the plane wave system (5.20) with the matrix $\boldsymbol{B}(\boldsymbol{d})$ from (5.22) given by

$$\boldsymbol{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & 0 \\ B_{13} & 0 & B_{22} \end{pmatrix}, \tag{7.29}$$

where

$$B_{11} = \lambda_L + 2\mu_L + (3\alpha + 2l_M + m_M)d_1, \tag{7.30}$$

$$B_{22} = \mu_L + \alpha d_1, \quad B_{12} = \alpha d_2, \quad B_{13} = \alpha d_3, \tag{7.31}$$

and $\alpha = \lambda_L + 2\mu_L + m_M$.

The eigenvalues of $\boldsymbol{B}$ are

$$\kappa_1 = \frac{1}{2}\left(B_{11} + B_{22} + \delta\right), \quad \kappa_2 = \frac{1}{2}\left(B_{11} + B_{22} - \delta\right), \quad \kappa_3 = B_{22}, \tag{7.32}$$

where

$$\delta = \sqrt{(B_{11} - B_{22})^2 + 4(B_{12}^2 + B_{13}^2)}. \tag{7.33}$$

The corresponding eigenvectors have components

$$\boldsymbol{q}_1 = \left[(B_{11} - B_{22} + \delta)/2\alpha, d_2, d_3\right], \tag{7.34}$$

$$\boldsymbol{q}_2 = \left[(B_{11} - B_{22} - \delta)/2\alpha, d_2, d_3\right], \tag{7.35}$$

$$\boldsymbol{q}_3 = \left[\,0, -d_3, d_2\right]. \tag{7.36}$$

Note that since $\boldsymbol{B}$ is symmetric it is not necessary to distinguish between its left and right eigenvectors, unlike the situation for $\boldsymbol{A}$.

Let us recall that from Lemma 5.11 we know that the pairs $\{\lambda_j, \boldsymbol{r}_j\}$ from the eigensystem of the whole matrix $\boldsymbol{A}$ are related to the pairs $\{\kappa_i, \boldsymbol{q}_i\}$ from the eigensystem of $\boldsymbol{B}$, specifically

$$\lambda_1 = -\sqrt{\kappa_1} = -\lambda_2, \qquad \lambda_3 = -\sqrt{\kappa_2} = -\lambda_4, \qquad \lambda_5 = -\sqrt{\kappa_3} = -\lambda_6,$$

$$\boldsymbol{r}_1 = \begin{bmatrix} -\sqrt{\kappa}_1 \boldsymbol{q}_1 \\ \boldsymbol{q}_1 \end{bmatrix}, \qquad \boldsymbol{r}_3 = \begin{bmatrix} -\sqrt{\kappa}_2 \boldsymbol{q}_2 \\ \boldsymbol{q}_2 \end{bmatrix}, \qquad \boldsymbol{r}_5 = \begin{bmatrix} -\sqrt{\kappa}_3 \boldsymbol{q}_3 \\ \boldsymbol{q}_3 \end{bmatrix},$$

$$\boldsymbol{r}_2 = \begin{bmatrix} \sqrt{\kappa}_1 \boldsymbol{q}_1 \\ \boldsymbol{q}_1 \end{bmatrix}, \qquad \boldsymbol{r}_4 = \begin{bmatrix} \sqrt{\kappa}_2 \boldsymbol{q}_2 \\ \boldsymbol{q}_2 \end{bmatrix}, \qquad \boldsymbol{r}_6 = \begin{bmatrix} \sqrt{\kappa}_3 \boldsymbol{q}_3 \\ \boldsymbol{q}_3 \end{bmatrix}.$$

Therefore, by taking into account the form of the eigenvectors $\boldsymbol{q}_i$, we conclude that the first pair $\{\kappa_1, \boldsymbol{q}_1\}$ corresponds to the pair of quasi-longitudinal waves characterized by the speeds $\pm\lambda_1$ and 'polarizations' $\boldsymbol{r}_1, \boldsymbol{r}_2$. The second pair $\{\kappa_2, \boldsymbol{q}_2\}$ corresponds to the quasi-shear waves propagating with the speeds $\pm\lambda_3$ and 'polarizations' $\boldsymbol{r}_3, \boldsymbol{r}_4$. Finally the third pair $\{\kappa_3, \boldsymbol{q}_3\}$ is related to a pair of pure shear waves whose speeds are $\pm\lambda_5$ and 'polarizations' $\boldsymbol{r}_5, \boldsymbol{r}_6$.

We now calculate the gradients of the eigenvalues with respect to the vector $\boldsymbol{d}$. We have, in component form,

$$D_{\boldsymbol{d}}\kappa_1 = \frac{1}{2}\left[4\alpha + 2l_M + m_M + \delta_{,d_1}, \delta_{,d_2}, \delta_{,d_3}\right], \tag{7.37}$$

$$D_{\boldsymbol{d}}\kappa_2 = \frac{1}{2}\left[4\alpha + 2l_M + m_M - \delta_{,d_1}, -\delta_{,d_2}, -\delta_{,d_3}\right], \tag{7.38}$$

$$D_{\boldsymbol{d}}\kappa_3 = [\alpha, 0, 0], \tag{7.39}$$

and we note that

$$D_{\boldsymbol{d}}\delta = \left[(B_{11} - B_{22})(2\alpha + 2l_M + m_M), 4\alpha^2 d_2, 4\alpha^2 d_3\right]/\delta. \tag{7.40}$$

We see immediately that

$$D_{\boldsymbol{d}}\kappa_3 \cdot \boldsymbol{q}_3 = 0 \quad \text{for any} \quad \boldsymbol{d}, \tag{7.41}$$

and hence the pure shear waves propagating with speeds λ_5 and λ_6 are *globally linearly degenerate*.

At $\boldsymbol{d} = \boldsymbol{0}$, we have also

$$(D_{\boldsymbol{d}}\kappa_2 \cdot \boldsymbol{q}_2)\big|_{\boldsymbol{d}=\boldsymbol{0}} = 0. \tag{7.42}$$

Thus, quasi-shear waves are *locally linearly degenerate*. It follows from (7.41) and (7.42) that the null condition provides no restrictions for shear and quasi–shear waves. On the other hand, it does impose restrictions on quasi-longitudinal waves. We have

$$(D_{\boldsymbol{d}}\kappa_1 \cdot \boldsymbol{q}_1)\big|_{\boldsymbol{d}=\boldsymbol{0}} = (3\alpha + 2l_M + m_M)\,(\lambda_L + \mu_L)/\alpha. \tag{7.43}$$

It follows from (7.43) that the null condition (7.28) for the Murnaghan material is given by

$$3\alpha + 2l_M + m_M \equiv 3(\lambda_L + 2\mu_L) + 2(l_M + 2m_M) = 0.$$

Here we are assuming that $\lambda_L + 2\mu_L > 0$. Thus, if the null condition is to be satisfied the material constants l_M and m_M must such that $l_M + 2m_M$ is negative.

7.3 Null Condition and Wave Self–interaction

Here we present the derivation of the null condition in yet another form, in terms of the nullity of wave self-interaction coefficients Γ^j_{jj}, which determine the magnitude of the interaction of the wave with itself. These coefficients are given by the formula

$$\Gamma^j_{jj} = \boldsymbol{l}_j \cdot (D_{\boldsymbol{w}}\boldsymbol{A}\boldsymbol{r}_j)\,\boldsymbol{r}_j, \tag{7.44}$$

where $\boldsymbol{l}_j$ and $\boldsymbol{r}_j$ are left and right eigenvectors of the matrix $\boldsymbol{A}$ in (12.6). Differentiation of the right-hand side of (7.44) in the direction $\boldsymbol{r}_j$ leads, after some algebra, to

$$\Gamma^j_{jj} = D_{\boldsymbol{w}}\lambda_j \cdot \boldsymbol{r}_j.$$

Hence, the wave self-interaction coefficient is equal to the coefficient that determines the genuine nonlinearity.

In [64] general formulas for wave interaction coefficients were obtained for arbitrary elastic plane waves and expressed in terms of derivatives of the strain energy. Following [64] we re-derive these formulas here, with attention focused on the self-interaction coefficients. From (5.26) and the fact that we assume the normalization $\boldsymbol{l}_i \cdot \boldsymbol{r}_j = \delta_{ij}$, we obtain the left eigenvectors of $\boldsymbol{A}$ as

$$\boldsymbol{l}_j(\boldsymbol{w}) = \left[(-1)^j \frac{\boldsymbol{q}_i}{2\sqrt{\kappa_i}}\,, \quad \frac{\boldsymbol{q}_i}{2} \right]^{\mathrm{T}}, \tag{7.45}$$

and we recall that $i = [\![(j+1)/2]\!]$.

We also have

$$D_{\boldsymbol{w}} \boldsymbol{A}(\boldsymbol{r}) = - \begin{bmatrix} \boldsymbol{0} & D_{\boldsymbol{d}} \boldsymbol{B}(\boldsymbol{q}) \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}$$

for any vector with six components $\boldsymbol{r} = [\boldsymbol{p}, \boldsymbol{q}]^{\mathrm{T}}$, where $\boldsymbol{q}$ is any vector with three components. Then, we find that the self-interaction coefficients are equal to

$$\Gamma^j_{jj} = \boldsymbol{l}_j \cdot (D_{\boldsymbol{w}} \boldsymbol{A} \boldsymbol{r}_j)\, \boldsymbol{r}_j = \frac{(-1)^j}{2\sqrt{\kappa_i}}\, \boldsymbol{q}_i \cdot (D_{\boldsymbol{d}} \boldsymbol{B}\, \boldsymbol{q}_i)\, \boldsymbol{q}_i. \tag{7.46}$$

Moreover, since $\boldsymbol{B} = D^2_{\boldsymbol{d}} V$, so (7.46) becomes

$$(-1)^{j+1} \Gamma^j_{jj} = \frac{1}{2\sqrt{\kappa_i}}\, D^3_{\boldsymbol{d}} V[\boldsymbol{q}_i, \boldsymbol{q}_i, \boldsymbol{q}_i], \tag{7.47}$$

where $D^3_{\boldsymbol{d}} V$ is a symmetric trilinear form. Thus, another way of expressing the *null condition* is as

$$D^3_{\boldsymbol{d}} V[\boldsymbol{q}_i, \boldsymbol{q}_i, \boldsymbol{q}_i] = 0 \quad \text{for } i = 1, 2, 3. \tag{7.48}$$

In the next section we express the self-interaction coefficients in terms of the strain energy $W(\boldsymbol{E})$ in order to obtain yet another general form of the null condition.

7.4 General Form of the Null Condition

In this section we derive a general form of the null condition, formulated as a condition that excludes plane wave self-interactions. We derive this condition for an arbitrary hyperelastic medium and for any direction of plane wave propagation.

First, we note that since

$$\boldsymbol{B} = \boldsymbol{Q}(\boldsymbol{k}) \equiv \mathcal{A}_{ijkl} k_j k_l,$$

it follows that

$$D_{\boldsymbol{d}}\boldsymbol{B} \equiv D^3_{\boldsymbol{d}}V = \mathcal{B}_{ijklmn}k_jk_lk_n,$$

where $\mathcal{B}_{ijklmn}$ is defined by (7.5). The null condition (7.48) can therefore be written as

$$\boldsymbol{\mathcal{B}}[\boldsymbol{q}_i \otimes \boldsymbol{k}, \boldsymbol{q}_i \otimes \boldsymbol{k}, \boldsymbol{q}_i \otimes \boldsymbol{k}] = 0.$$

We may also write this in terms of derivatives of W with respect to $\boldsymbol{E}$. The result is

$$D^3_{\boldsymbol{E}}W[\boldsymbol{k}\otimes\boldsymbol{Q}_i, \boldsymbol{k}\otimes\boldsymbol{Q}_i, \boldsymbol{k}\otimes\boldsymbol{Q}_i] + 3D^2_{\boldsymbol{E}}W[\boldsymbol{k}\otimes\boldsymbol{k}, \boldsymbol{k}\otimes\boldsymbol{Q}_i]\,\boldsymbol{q}_i\cdot\boldsymbol{q}_i = 0, \quad (7.49)$$

where $\boldsymbol{Q}_i = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{q}_i$.

To obtain this result we have used the formulas (given in index notation)

$$\frac{\partial E_{pq}}{\partial F_{ij}} = \frac{1}{2}(\delta_{jp}F_{iq} + \delta_{jp}F_{ip}),$$

and

$$(D_{\boldsymbol{F}}W)_{ij} = F_{ip}(D_{\boldsymbol{E}}W)_{pj}, \quad (7.50)$$

$$(D^2_{\boldsymbol{F}}W)_{ijkl} = F_{ip}F_{kq}(D^2_{\boldsymbol{E}}W)_{jplq} + (D_{\boldsymbol{E}}W)_{jl}\delta_{ik}, \quad (7.51)$$

$$\begin{aligned}(D^3_{\boldsymbol{F}}W)_{ijklmn} = {}& F_{ip}F_{kq}F_{mr}(D^3_{\boldsymbol{E}}W)_{jplqnr} + (D^2_{\boldsymbol{E}}W)_{jnlp}F_{kp}\delta_{im}\\ &+ (D^2_{\boldsymbol{E}}W)_{jpln}F_{ip}\delta_{km} + (D^2_{\boldsymbol{E}}W)_{jlnp}F_{mp}\delta_{ik}. \quad (7.52)\end{aligned}$$

For an isotropic material, the acoustic tensor and the null condition can be calculated explicitly using (7.8) and (7.9), respectively. Alternatively, if W is treated as a function of $\boldsymbol{E}$ then we use the connections (7.51) and (7.52) together with the expansions

$$(D_{\boldsymbol{E}}W)_{ij} = \sum_{p=1}^{3} W_p(D_{\boldsymbol{E}}I_p)_{ij},$$

$$(D^2_{\boldsymbol{E}}W)_{ijkl} = \sum_{p,q=1}^{3} W_{pq}(D_{\boldsymbol{E}}I_p)_{ij}(D_{\boldsymbol{E}}I_q)_{kl} + \sum_{p=1}^{3} W_p(D^2_{\boldsymbol{E}}I_p)_{ijkl},$$

$$
\begin{aligned}
(D^3_{\boldsymbol{E}}W)_{ijklmn} = & \sum_{p,q,r=1}^{3} W_{pqr}(D_{\boldsymbol{E}}I_p)_{ij}(D_{\boldsymbol{E}}I_q)_{kl}(D_{\boldsymbol{E}}I_r)_{mn} \\
& + \sum_{p,q=1}^{3} W_{pq}(D_{\boldsymbol{E}}I_p)_{ij}(D^2_{\boldsymbol{E}}I_q)_{klmn} \\
& + \sum_{p,q=1}^{3} W_{pq}(D_{\boldsymbol{E}}I_p)_{kl}(D^2_{\boldsymbol{E}}I_q)_{ijmn} \\
& + \sum_{p,q=1}^{3} W_{pq}(D_{\boldsymbol{E}}I_p)_{mn}(D^2_{\boldsymbol{E}}I_q)_{ijkl} \\
& + \sum_{p=1}^{3} W_p(D^3_{\boldsymbol{E}}I_p)_{ijklmn}.
\end{aligned}
$$

Additionally, we need the formulas

$$
D_{\boldsymbol{E}}I_1 = \boldsymbol{I}, \quad D_{\boldsymbol{E}}I_2 = I_1\boldsymbol{I} - \boldsymbol{E}, \quad D_{\boldsymbol{E}}I_3 = \boldsymbol{E}^2 - I_1\boldsymbol{E} + I_2\boldsymbol{I},
$$

$$
D^2_{\boldsymbol{E}}I_1 = \boldsymbol{O}, \quad D^2_{\boldsymbol{E}}I_2 = \boldsymbol{I} \otimes \boldsymbol{I} - \boldsymbol{\mathcal{I}},
$$

and

$$
\frac{\partial^2 I_3}{\partial \boldsymbol{E} \partial \boldsymbol{E}} = 2\boldsymbol{\mathcal{J}}\boldsymbol{E} - (\boldsymbol{I} \otimes \boldsymbol{E} + \boldsymbol{E} \otimes \boldsymbol{I}) + I_1(\boldsymbol{I} \otimes \boldsymbol{I} - \boldsymbol{\mathcal{I}}),
$$

where, in index notation,

$$
(\boldsymbol{I} \otimes \boldsymbol{E})_{ijkl} = \delta_{ij}E_{kl}, \quad \mathcal{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),
$$

$$
\mathcal{J}_{ijklmn} = \frac{1}{4}(\delta_{ik}\mathcal{I}_{jlmn} + \delta_{il}\mathcal{I}_{jkmn} + \delta_{im}\mathcal{I}_{kljn} + \delta_{in}\mathcal{I}_{kljm}),
$$

and

$$
\boldsymbol{\mathcal{J}}\boldsymbol{E} = \mathcal{J}_{ijklmn}E_{mn} = \frac{1}{2}(\delta_{ik}E_{jl} + \delta_{il}E_{jk} + \delta_{jl}E_{ik} + \delta_{jk}E_{il}).
$$

Finally, the third derivatives of I_1 and I_2 vanish and, in index notation,

$$
(D^3_{\boldsymbol{E}}I_3)_{ijklmn} = 2\mathcal{J}_{ijklmn} + \delta_{ij}\delta_{kl}\delta_{mn} - (\delta_{ij}\mathcal{I}_{klmn} + \delta_{kl}\mathcal{I}_{ijmn} + \delta_{mn}\mathcal{I}_{ijkl}).
$$

The resulting expressions for the acoustic tensor and the null condition are very lengthy in this form and are not therefore listed here in the general case.

7.4.1 General Null Condition for the Murnaghan Material

The general formula (7.49) obtained for the null condition can also, because of the symmetry of $\boldsymbol{E}$, be expressed by replacing $\boldsymbol{k} \otimes \boldsymbol{Q}_i$ by $\boldsymbol{Q}_i \otimes \boldsymbol{k}$ or by its symmetric part. When evaluated in the reference configuration the null condition (7.49) becomes

$$\left[\left(2W^0_{111} + 3W^0_{12}\right)(\boldsymbol{q}_i \cdot \boldsymbol{k})^2 + 3\left(2W^0_{11} - W^0_{12}\right)(\boldsymbol{q}_i \cdot \boldsymbol{q}_i)\right](\boldsymbol{q}_i \cdot \boldsymbol{n}) = 0, \quad (7.53)$$

where we have now written $\boldsymbol{q}_i = \boldsymbol{Q}_i, \boldsymbol{n} = \boldsymbol{k}$. This is the general form of the null condition for an isotropic elastic material. It can be seen immediately that for transverse waves ($\boldsymbol{q}_i \cdot \boldsymbol{n} = 0$) no restriction is imposed by the null condition. For pure longitudinal waves with $\boldsymbol{q}_i = \boldsymbol{n}$ and $\boldsymbol{n} \cdot \boldsymbol{n} = 1$, on the other hand, the first factor in (7.53) yields the restriction $W^0_{111} + 3W^0_{11} = 0$, thus recovering the result (7.22).

For the Murnaghan material the first factor in (7.53) reduces the null condition to

$$(2l_M + m_M)(\boldsymbol{n} \cdot \boldsymbol{q}_i)^2 + 3\alpha(\boldsymbol{q}_i \cdot \boldsymbol{q}_i) = 0,$$

where we recall that $\alpha = \lambda_L + 2\mu_L + m_M$. For $\boldsymbol{q}_i = \boldsymbol{n}$ this becomes $2l_M + m_M + 3\alpha = 0$. Hence the null condition for the Murnaghan material is

$$3(\lambda_L + 2\mu_L) + 2(l_M + 2m_M)$$

Remark 7.4. For the St. Venant-Kirchhoff material, for which the strain-energy function is given by (see, e.g., Ciarlet [32], p. 155)

$$W = \frac{1}{2}(\lambda_L + 2\mu_L)I_1^2 - 2\mu_L I_2, \quad (7.54)$$

the corresponding general null condition is found to be

$$(\lambda_L + 2\mu_L)(\boldsymbol{q}_i \cdot \boldsymbol{q}_i)(\boldsymbol{n} \cdot \boldsymbol{q}_i) = 0.$$

In particular, for the longitudinal wave with $\boldsymbol{q}_i = \boldsymbol{n}$, this yields

$$\lambda_L + 2\mu_L = 0,$$

which contradicts the usual assumptions adopted for the Lamé moduli, one of which is $\lambda_L + 2\mu_L > 0$. This follows from the strong ellipticity condition in the reference configuration, which we assume to hold.

7.5 Application to a Class of Strain Energy Functions

Here we show an example of a compressible elastic material which may or may not satisfy the null condition, depending on the choice of a parameter which characterizes the strain energy function for this material. In the literature there are many examples of isotropic strain-energy functions for *incompressible* elastic materials, but far fewer for *compressible* materials. Amongst the available compressible energy functions are the classes introduced by Jiang and Ogden [88]. These include energy functions of the form

$$W = \bar{W}(\bar{I}_1, \bar{I}_3) = \bar{f}(\bar{I}_1)h_1(\bar{I}_3) + h_2(\bar{I}_3), \tag{7.55}$$

where the overbars indicate that the invariants are principal invariants of $\boldsymbol{C}$, not $\boldsymbol{E}$, with $\bar{f}$ a function of $\bar{I}_1$ and h_1 and h_2 functions of $\bar{I}_3$ that satisfy certain conditions which we do not specify here.

We note the connections

$$\bar{I}_1 = 3 + 2I_1, \quad \bar{I}_3 = 1 + 2I_1 + 4I_2 + 8I_3, \tag{7.56}$$

which will be used below.

For simplicity of illustration we specialize (7.55) by setting $\bar{f}(\bar{I}_1) = f(I_1)$, $h_1 \equiv 1$, $h_2 = h$, so that it becomes

$$W = W(I_1, I_2, I_3) = f(I_1) + h(\bar{I}_3), \tag{7.57}$$

in which $(7.56)_2$ is required.

For the energy and the stress to vanish in the reference configuration we require, recalling (7.20),

$$W^0 = f(0) + h(1) = 0, \quad W_1^0 = f'(0) + 2h'(1) = 0, \tag{7.58}$$

while for compatibility with the classical theory we must have

$$4h'(1) = -2\mu_L = W_2^0, \quad W_{11}^0 = f''(0) + 4h''(1) = \lambda_L + 2\mu_L, \tag{7.59}$$

from which we deduce $f'(0) = \mu_L$.

We also calculate $W_{111}^0 = f'''(0) + 8h'''(1)$, so that the null condition (7.22) becomes

$$3(\lambda_L + 2\mu_L) + f'''(0) + 8h'''(1) = 0. \tag{7.60}$$

Jiang and Ogden [88] considered, in particular, functions of the form

$$\bar{f}(\bar{I}_1) = \frac{\mu_L}{k}\left(\frac{\bar{I}_1 - 1}{2}\right)^k \quad \longrightarrow \quad f(I_1) = \frac{\mu_L}{k}(I_1 + 1)^k\,, \tag{7.61}$$

where $k \geq 1/2$ is a material parameter, f satisfies

$$f(0) = \frac{\mu_L}{k}, \quad f'(0) = \mu_L, \quad f''(0) = (k-1)\mu_L, \tag{7.62}$$

and h is restricted according to

$$h(1) = -\frac{\mu_L}{k}, \quad h'(1) = -\frac{1}{2}\mu_L, \quad h''(1) = \lambda_L + (3-k)\mu_L. \tag{7.63}$$

For the considered specialization equation (7.60) becomes

$$3(\lambda_L + 2\mu_L) + \mu_L(k-1)(k-2) + 8h'''(1) = 0. \tag{7.64}$$

Subject to (7.63) there is considerable flexibility in the choice of the form of h, in particular in the value of $h'''(1)$. Thus, it is possible to find energy functions within the considered class for which the null condition is either satisfied or not satisfied, and this comment applies to wide ranges of other possible energy functions within different classes.

7.6 Some Remarks

We have found expressions for the null condition in several different forms. These have been derived for an arbitrary hyperelastic medium and for any plane wave direction irrespective of the use of the invariants or the assumption of isotropy. They can therefore be useful for application to anisotropic materials, although we do not claim that the null condition alone suffices to prove global existence and uniqueness in the general anisotropic case.

We have derived explicit restrictions imposed by the null condition for the Murnagham material in terms of the second- and third-order elastic constants (Lamé and Murnaghan constants). We have also noted that for the St. Venant-Kirchhoff material the null condition is too strong in that it contradicts the inequality $\lambda_L + 2\mu_L > 0$ and would therefore exclude self-interaction of longitudinal waves and *a fortiori* the propagation of longitudinal waves in the reference configuration. Finally, we have examined the form of the null condition for a special class of energy functions, within which particular energy functions can be constructed that either do or do not satisfy the null condition.

8

Interaction Coefficients

8.1 Introduction

In this section we derive general formulas for the interaction coefficients for plane elastic waves. The formulas are expressed entirely in terms of the elastic strain energy function. The formulas are valid for any direction of the elastic plane wave propagation and for arbitrary anisotropy of the medium. This is one of the main results of our work which was obtained using only simple properties of the Fréchet derivative.

We recall the definition of the *interaction coefficients* Γ^j_{pq} for an arbitrary hyperbolic plane waves system:

$$\boldsymbol{w}_{,t} + \boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k})\boldsymbol{w}_{,x} = \boldsymbol{0}. \tag{8.1}$$

Definition 8.1. *(Interaction coefficients). The interaction coefficients are defined as*

$$\Gamma^j_{pq} = \boldsymbol{l}_j \cdot (D_{\boldsymbol{w}}\, \boldsymbol{A}\boldsymbol{r}_p)\boldsymbol{r}_q \tag{8.2}$$

with $\boldsymbol{l}_j$ *and* $\boldsymbol{r}_p, \boldsymbol{r}_q$ *the appropriate left and right eigenvectors of the matrix* $\boldsymbol{A}$, *and* $D_{\boldsymbol{w}}\boldsymbol{A}$ *is a Fréchet derivative of* $\boldsymbol{A}$ *with respect to* $\boldsymbol{w}$.

Assume that

$$\boldsymbol{A} = D_{\boldsymbol{w}}\boldsymbol{f}.$$

Then

$$D_{\boldsymbol{w}}\boldsymbol{A} = D^2_{\boldsymbol{w}}\, \boldsymbol{f},$$

so the interaction coefficients for systems derived from conservation laws are symmetric,

$$\Gamma^j_{pq} = \boldsymbol{l}_j \cdot D^2_{\boldsymbol{w}} \, \boldsymbol{f}(\boldsymbol{r}_p, \boldsymbol{r}_q) = \boldsymbol{l}_j \cdot D^2_{\boldsymbol{w}} \, \boldsymbol{f}(\boldsymbol{r}_q, \boldsymbol{r}_p) = \Gamma^j_{qp}. \tag{8.3}$$

We will calculate and interpret these coefficients for systems of plane elasticity. Let us recall that in case of elasticity

$$\boldsymbol{f}(\boldsymbol{w}) = \begin{bmatrix} \boldsymbol{s}(\boldsymbol{d}) \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} D_{\boldsymbol{d}} V \\ \boldsymbol{v} \end{bmatrix},$$

hence the interaction coefficients for elastic waves are also symmetric (c.f. Sec. 3.3).

8.2 General Form of the Interaction Coefficients for Plane Elastic Waves

In order to express the interaction coefficients in terms of the eigensystem of the matrix $\boldsymbol{B}(\boldsymbol{d})$ from (5.22), we will make use of the Lemma 7.2. We have

$$D_{\boldsymbol{w}} \, \boldsymbol{A}(\boldsymbol{r}) = - \begin{bmatrix} \boldsymbol{0} & D_{\boldsymbol{d}} \boldsymbol{B}(\boldsymbol{q}) \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

where $\boldsymbol{q}$ is the eigenvector of $\boldsymbol{B}$ which is the second part of $\boldsymbol{r}$.

Now using the Lemma 7.2, we calculate the interaction coefficients to be

$$\begin{aligned} \Gamma^j_{pq} &= \boldsymbol{l}_j \cdot (D_{\boldsymbol{w}} \, \boldsymbol{A} \boldsymbol{r}_p) \boldsymbol{r}_q \\ &= (-1)^j \frac{1}{2\sqrt{\kappa_{j'}}} \, \boldsymbol{q}_{j'} \cdot (D_{\boldsymbol{d}} \, \boldsymbol{B} \boldsymbol{q}_{p'}) \boldsymbol{q}_{q'} \end{aligned} \tag{8.4}$$

and we recall that $j' = [\![(j+1)/2]\!], p' = [\![(p+1)/2]\!], q' = [\![(q+1)/2]\!],$.

Moreover, according to (5.22), we have $\boldsymbol{B} = D^2_{\boldsymbol{d}} V$, so that (8.4) becomes

$$\Gamma^j_{pq} = (-1)^j \frac{1}{2\sqrt{\kappa_{j'}}} \, D^3_{\boldsymbol{d}} V(\boldsymbol{q}_{j'}, \boldsymbol{q}_{p'}, \boldsymbol{q}_{q'}), \tag{8.5}$$

where $D^3_{\boldsymbol{d}} V$ is a symmetric trilinear form.

Differentiating (5.22) with respect to $\boldsymbol{d}$, using (5.17), (6.1), the chain and Leibniz rules, we get

$$\begin{aligned} D_{\boldsymbol{d}} \, \boldsymbol{B} = D^3_{\boldsymbol{d}} V \; &= \; D^3_{\boldsymbol{E}} W \diamond D_{\boldsymbol{F}} \, \boldsymbol{E} \cdot D_{\boldsymbol{d}} \, \boldsymbol{F} \\ &+ \; 3 \, D^2_{\boldsymbol{E}} W (D^2_{\boldsymbol{F}} \, \boldsymbol{E} \circ D_{\boldsymbol{d}} \, \boldsymbol{F}, D_{\boldsymbol{F}} \, \boldsymbol{E} \cdot D_{\boldsymbol{d}} \, \boldsymbol{F}). \end{aligned} \tag{8.6}$$

where $\diamond$ denotes the following operation:

$$D^3_{\boldsymbol{E}} W \diamond \boldsymbol{L}(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3) = D^3_{\boldsymbol{E}} W(\boldsymbol{L}\boldsymbol{b}_1, \boldsymbol{L}\boldsymbol{b}_2, \boldsymbol{L}\boldsymbol{b}_3) \tag{8.7}$$

for any operator $\boldsymbol{L}$. Equation (8.6) shows the effects of geometric versus physical nonlinearity. If we assume the material is physically linear, then the term $D^3_{\boldsymbol{E}} W$ vanishes, whereas if we assume it is geometrically linear, then $D^2_{\boldsymbol{F}} \boldsymbol{E} = \boldsymbol{0}$ so the second term vanishes. In particular, if we assume both linearities, all coefficients vanish and the system is linear. In order to capture the fully nonlinear behavior, we assume both physical and geometric nonlinearities.

Using (5.32), (5.33) and (5.34), with the help of (5.28) and (8.7), we get from (8.5) and (8.6):

$$\Gamma^j_{pq} = \frac{1}{2\lambda_j} \left\{ D^3_{\boldsymbol{E}} W(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_{j'}}, \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_{p'}}, \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_{q'}}) + D^2_{\boldsymbol{E}} W(\boldsymbol{\mathcal{Q}}_{j'p'q'}, \boldsymbol{k} \otimes \boldsymbol{k}) \right\}, \tag{8.8}$$

where

$$\lambda_j = (-1)^j \sqrt{\kappa_i},$$

$$\boldsymbol{\mathcal{Q}}_{j'p'q'} = (\boldsymbol{q}_{p'} \cdot \boldsymbol{q}_{q'})\, \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_{j'}} + (\boldsymbol{q}_{q'} \cdot \boldsymbol{q}_{j'})\, \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_{p'}} + (\boldsymbol{q}_{j'} \cdot \boldsymbol{q}_{p'})\, \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_{q'}}, \tag{8.9}$$

and $\boldsymbol{\mathcal{P}}_{\boldsymbol{q}}$ is defined (see (5.33)) as

$$\boldsymbol{\mathcal{P}}_{\boldsymbol{q}} = \frac{1}{2} (\boldsymbol{q} \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \boldsymbol{q}) + (\boldsymbol{d} \cdot \boldsymbol{q})\, \boldsymbol{k} \otimes \boldsymbol{k}. \tag{8.10}$$

This last formula is true when the gradient of a homogeneous deformation $\boldsymbol{\pi} = \boldsymbol{I}$.

We now have an explicit expression for the interaction coefficients in an arbitrary medium and for any direction of plane waves propagation. Once a strain energy function describing the material is given, we first calculate the matrix $\boldsymbol{B}(\boldsymbol{d}, \boldsymbol{k})$ from (5.22), and then we find its eigensystem. Finally, we directly substitute the eigenvalues and eigenvectors into (8.8) to obtain the coefficients. Please note that we need to find the eigensystem of the 3×3 matrix $\boldsymbol{B}(\boldsymbol{d}, \boldsymbol{k})$, instead of the eigensystem of the full 6×6 matrix $\boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k})$.

8.3 Interaction Coefficients for a General Isotropic Medium

We now carry out the calculations described above, applying them to isotropic materials by introducing the explicit form of the strain energy function $W(\boldsymbol{E})$ into the equations of motion. For simplicity, we assume that the background homogeneous deformation is trivial,

$$\boldsymbol{\pi} = \boldsymbol{I},$$

that is the plane waves propagate on an undisturbed background state.

Besides the objectivity assumption, the strain energy function $W(\boldsymbol{F})$ for isotropic materials must satisfy also the condition

$$W(\boldsymbol{F}) = W(\boldsymbol{F}\boldsymbol{Q}^T) \tag{8.11}$$

for any $\boldsymbol{Q} \in SO(3)$. The objectivity requirement and (8.11) together lead to the formula

$$W(\boldsymbol{E}) = W(\boldsymbol{Q}\boldsymbol{E}\boldsymbol{Q}^T) \tag{8.12}$$

for any fixed proper orthogonal matrix $\boldsymbol{Q}$. Equivalently the isotropic material may be characterized by the requirement that the energy function W depends only on *three* invariants $I_{\boldsymbol{E}}$, $II_{\boldsymbol{E}}$ and $III_{\boldsymbol{E}}$ of the strain matrix $\mathbf{E}$,

$$W(\boldsymbol{E}) = \widetilde{W}(\boldsymbol{J}_{\boldsymbol{E}}) \quad \text{where} \quad \boldsymbol{J}_{\boldsymbol{E}} = \{I_{\boldsymbol{E}}, II_{\boldsymbol{E}}, III_{\boldsymbol{E}}\}. \tag{8.13}$$

As we have already mentioned before, these strain invariants are given by the formulas:

$$\begin{aligned} I_{\boldsymbol{E}} &= tr\ \boldsymbol{E} \\ II_{\boldsymbol{E}} &= \frac{1}{2}((tr\ \boldsymbol{E})^2 - tr\ (\boldsymbol{E}^2)) \\ III_{\boldsymbol{E}} &= det\ \boldsymbol{E}. \end{aligned} \tag{8.14}$$

Expanding the energy $\widetilde{W}(\boldsymbol{J}_{\boldsymbol{E}})$ in terms of the invariants in the vicinity of the undeformed state, we get

$$\begin{aligned} \widetilde{W}(\boldsymbol{J}_{\boldsymbol{E}}) = \frac{\partial \widetilde{W}}{\partial I_{\boldsymbol{E}}} I_{\boldsymbol{E}} + \frac{\partial \widetilde{W}}{\partial II_{\boldsymbol{E}}} II_{\boldsymbol{E}} + \frac{1}{2}\frac{\partial^2 \widetilde{W}}{\partial {I_{\boldsymbol{E}}}^2} {I_{\boldsymbol{E}}}^2 \\ + \frac{\partial \widetilde{W}}{\partial III_{\boldsymbol{E}}} III_{\boldsymbol{E}} + \frac{1}{2}\frac{\partial^2 \widetilde{W}}{\partial I_{\boldsymbol{E}} \partial II_{\boldsymbol{E}}} I_{\boldsymbol{E}} II_{\boldsymbol{E}} + \frac{1}{6}\frac{\partial^3 \widetilde{W}}{\partial {I_{\boldsymbol{E}}}^3} {I_{\boldsymbol{E}}}^3 + ... \end{aligned}$$

We assume that there is no strain in the undeformed constant state. The derivatives of energy with respect to strain invariants, evaluated at the zero constant state, are given by the elastic constants as follows:

$$\frac{\partial \widetilde{W}}{\partial I_{\boldsymbol{E}}} = 0, \qquad \frac{\partial \widetilde{W}}{\partial II_{\boldsymbol{E}}} = -2\mu_L, \qquad \frac{\partial^2 \widetilde{W}}{\partial {I_{\boldsymbol{E}}}^2} = \lambda_L + 2\mu_L,$$
$$\frac{\partial \widetilde{W}}{\partial III_{\boldsymbol{E}}} = n_M, \qquad \frac{\partial^2 \widetilde{W}}{\partial I_{\boldsymbol{E}} \partial II_{\boldsymbol{E}}} = -4m_M, \qquad \frac{\partial^3 \widetilde{W}}{\partial {I_{\boldsymbol{E}}}^3} = 2l_M + 4m_M.$$

Here λ_L and μ_L are the second order Lame constants, and l_M, m_M, and n_M are third order Murnaghan constants. In this way, we get the Murnaghan [126] expression for the strain energy of an isotropic solid with third order terms:

$$\widetilde{W}(\boldsymbol{J}_{\boldsymbol{E}}) = \frac{1}{2}(\lambda_L + 2\mu_L){I_{\boldsymbol{E}}}^2 - 2\mu_L II_{\boldsymbol{E}} + \frac{1}{3}(l_M + 2m_M){I_{\boldsymbol{E}}}^3 - 2m_M I_{\boldsymbol{E}} II_{\boldsymbol{E}} + n_M III_{\boldsymbol{E}}. \tag{8.15}$$

Since our calculations involve derivatives up to the third order, and we are expanding about the zero constant state, we regard this as the general form of an isotropic material. We can now invoke the chain rule again to calculate the derivatives $D_{\boldsymbol{E}} W$ in terms of $D_{\mathbf{I_E}} \widetilde{W}$ and $D_{\boldsymbol{E}} \boldsymbol{I}_{\boldsymbol{E}}$.

8.3.1 Interaction Coefficients for a Physically Linear Material

Matrix $\boldsymbol{B}$ for a Physically Linear Material

We provide some details, first assuming that the medium is geometrically nonlinear but restricting ourselves to the *St. Venant–Kirchhoff* form of the energy function, that is taking into account only the quadratic terms in (8.15),

$$\widetilde{W}(\boldsymbol{J}_{\boldsymbol{E}}) = \frac{1}{2}(\lambda_L + 2\mu_L){I_{\boldsymbol{E}}}^2 - 2\mu_L II_{\boldsymbol{E}}. \tag{8.16}$$

A short calculation shows that for an arbitrary second order tensor $\boldsymbol{P}$, we have

$$\begin{aligned} (D_{\boldsymbol{E}} I_{\boldsymbol{E}})(\boldsymbol{P}) &= I_{\boldsymbol{P}} \equiv tr\, \boldsymbol{P}, \\ (D_{\boldsymbol{E}} II_{\boldsymbol{E}})(\boldsymbol{P}) &= tr\, \boldsymbol{E}\; tr\, \boldsymbol{P} - tr\,(\boldsymbol{E}\,\boldsymbol{P}) \quad \text{and} \\ (D^2_{\boldsymbol{E}}\, II_{\boldsymbol{E}})(\boldsymbol{P}_1, \boldsymbol{P}_2) &= tr\, \boldsymbol{P}_1\; tr\, \boldsymbol{P}_2 - tr\,(\boldsymbol{P}_1\, \boldsymbol{P}_2), \end{aligned} \tag{8.17}$$

so that for the energy given by (8.16),

$$\begin{aligned} (D_{\boldsymbol{E}} \widetilde{W})(\boldsymbol{P}) &= \lambda_L\; tr\, \boldsymbol{E}\; tr\, \boldsymbol{P} + 2\,\mu_L\; tr\,(\boldsymbol{E}\,\boldsymbol{P}) \quad \text{and} \\ (D^2_{\boldsymbol{E}}\, \widetilde{W})(\boldsymbol{P}_1, \boldsymbol{P}_2) &= \lambda_L\; tr\, \boldsymbol{P}_1\; tr\, \boldsymbol{P}_2 + 2\,\mu_L\; tr\,(\boldsymbol{P}_1\, \boldsymbol{P}_2). \end{aligned} \tag{8.18}$$

We are now in a position to write down the matrix $\boldsymbol{B}(\boldsymbol{d},\boldsymbol{k})$. From (5.35) we have

$$\begin{aligned}\boldsymbol{q}_1\cdot\boldsymbol{B}\boldsymbol{q}_2 &= D^2_{\boldsymbol{E}}W(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_1},\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_2})+(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)\,D_{\boldsymbol{E}}W(\boldsymbol{k}\otimes\boldsymbol{k})\\ &= \lambda_M\,tr\,\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_1}tr\,\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_2}+2\mu_M\,tr\,(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_1}\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_2})+\\ &\quad(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)\,[\lambda_L\,tr\,\boldsymbol{E}\,tr\,(\boldsymbol{k}\otimes\boldsymbol{k})+2\mu_L\,tr\,(\boldsymbol{E}(\boldsymbol{k}\otimes\boldsymbol{k}))]\,.\end{aligned}$$

Recalling that $\boldsymbol{\pi}=\boldsymbol{I}$ and $\boldsymbol{k}\cdot\boldsymbol{k}=1$, we get that

$$tr\,\boldsymbol{\mathcal{P}}_{\boldsymbol{q}}=\boldsymbol{q}\cdot\boldsymbol{k}+\boldsymbol{d}\cdot\boldsymbol{q}, \tag{8.19}$$

and

$$\begin{aligned}tr\,(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_1}\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_2}) &= \tfrac{1}{2}(\boldsymbol{q}_1\cdot\boldsymbol{k})\,(\boldsymbol{k}\cdot\boldsymbol{q}_2)+\tfrac{1}{2}(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)+(\boldsymbol{q}_1\cdot\boldsymbol{k})\,(\boldsymbol{d}\cdot\boldsymbol{q}_2)+\\ &\quad(\boldsymbol{q}_2\cdot\boldsymbol{k})\,(\boldsymbol{d}\cdot\boldsymbol{q}_1)+(\boldsymbol{q}_1\cdot\boldsymbol{d})\,(\boldsymbol{d}\cdot\boldsymbol{q}_2).\end{aligned} \tag{8.20}$$

Moreover

$$tr\,\boldsymbol{E}\,tr\,(\boldsymbol{k}\otimes\boldsymbol{k})=tr\,(\boldsymbol{E}(\boldsymbol{k}\otimes\boldsymbol{k}))=\boldsymbol{d}\cdot\boldsymbol{k}+\frac{1}{2}\boldsymbol{d}\cdot\boldsymbol{d}.$$

Hence

$$\begin{aligned}\boldsymbol{q}_1\cdot\boldsymbol{B}\,q_2 &= (\lambda_L+\mu_L)\,(\boldsymbol{q}_1\cdot\boldsymbol{k})\,(\boldsymbol{k}\cdot\boldsymbol{q}_2)+\mu_L(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)+\\ &\quad(\lambda_L+2\mu_L)\,[(\boldsymbol{q}_1\cdot\boldsymbol{k})\,(\boldsymbol{d}\cdot\boldsymbol{q}_2)+(\boldsymbol{q}_2\cdot\boldsymbol{k})\,(\boldsymbol{d}\cdot\boldsymbol{q}_1)+\\ &\quad(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)\,(\boldsymbol{d}\cdot\boldsymbol{k})+(\boldsymbol{q}_1\cdot\boldsymbol{d})\,(\boldsymbol{d}\cdot\boldsymbol{q}_2)+\frac{1}{2}(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)\,(\boldsymbol{d}\cdot\boldsymbol{d})],\end{aligned}$$

so that

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{d},\boldsymbol{k}) &= (\lambda_L+\mu_L)\,\boldsymbol{k}\otimes\boldsymbol{k}+\mu_L\,\boldsymbol{I}+(\lambda_L+2\,\mu_L)\,[\boldsymbol{k}\otimes\boldsymbol{d}+\\ &\quad\boldsymbol{d}\otimes\boldsymbol{k}+(\boldsymbol{d}\cdot\boldsymbol{k})\,\boldsymbol{I}+\boldsymbol{d}\otimes\boldsymbol{d}+\frac{1}{2}(\boldsymbol{d}\cdot\boldsymbol{d})\,\boldsymbol{I}].\end{aligned} \tag{8.21}$$

Interaction Coefficients for $\boldsymbol{d}=\boldsymbol{0}$

We will now derive the formula for the interaction coefficients in the unperturbed constant state $\boldsymbol{d}=\boldsymbol{0}$.

We have from (8.21)

$$\boldsymbol{q}_1\cdot\boldsymbol{B}(\boldsymbol{0})\,\boldsymbol{q}_2=(\lambda_L+\mu_L)\,(\boldsymbol{q}_1\cdot\boldsymbol{k})\,(\boldsymbol{k}\cdot\boldsymbol{q}_2)+\mu_L\,(\boldsymbol{q}_1\cdot\boldsymbol{q}_2),$$

so that

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{0},\boldsymbol{k}) &= (\lambda_L+\mu_L)\,\boldsymbol{k}\otimes\boldsymbol{k}+\mu_L\,\boldsymbol{I} \\ &= (\lambda_L+2\,\mu_L)\,\boldsymbol{k}\otimes\boldsymbol{k}+\mu_L\,(\boldsymbol{I}-\boldsymbol{k}\otimes\boldsymbol{k}). \end{aligned} \tag{8.22}$$

As this is a decomposition into orthogonal projections, the eigenvalues of $\boldsymbol{B}(\boldsymbol{0})$ are $\lambda_L + 2\,\mu_L \equiv c_l^2$ and $\mu_L \equiv c_s^2$ (of multiplicity two); c_l and c_s denote the speeds of linearized longitudinal and shear waves respectively under the assumption that $\rho_0 = 0$. The associated eigenvectors are $\boldsymbol{k}$, and $\boldsymbol{k}_1^{\perp}$, $\boldsymbol{k}_2^{\perp}$ respectively. Here $\boldsymbol{k}_1^{\perp}$ and $\boldsymbol{k}_2^{\perp}$ are unit vectors orthogonal to $\boldsymbol{k}$.

To calculate the coefficients we substitute the quantities $\boldsymbol{\mathcal{P}}_{\boldsymbol{q}}$ and $\boldsymbol{\mathcal{Q}}_{jpq}$ from (8.10) and (8.9), and plug these into (8.8). Here the $\boldsymbol{q}$'s, being eigenvectors of $\boldsymbol{B}$, take on values $\boldsymbol{k}$, $\boldsymbol{k}_1^{\perp}$ or $\boldsymbol{k}_2^{\perp}$. For the St. Venant–Kirchhoff material, using (8.18) in (8.8) and simplifying, we get (no sum over repeated indices)

$$\begin{aligned} 2\,\lambda_j\,\Gamma^j_{pq} &= D^2_{\boldsymbol{E}}\,W(\boldsymbol{\mathcal{Q}}_{j'p'q'},\boldsymbol{k}\otimes\boldsymbol{k}) \\ &= \lambda_L\;tr\,\boldsymbol{\mathcal{Q}}_{j'p'q'}tr\,(\boldsymbol{k}\otimes\boldsymbol{k})+2\,\mu_L\;tr\,(\boldsymbol{\mathcal{Q}}_{j'p'q'}\boldsymbol{k}\otimes\boldsymbol{k}) \\ &= (\lambda_L+2\,\mu_L)\;\mathcal{D}_{j'p'q'}(\boldsymbol{k}), \end{aligned} \tag{8.23}$$

where

$$\mathcal{D}_{j'p'q'}(\boldsymbol{k}) \equiv (\boldsymbol{q}_{j'}\cdot\boldsymbol{k})(\boldsymbol{q}_{p'}\cdot\boldsymbol{q}_{q'})+(\boldsymbol{q}_{p'}\cdot\boldsymbol{k})(\boldsymbol{q}_{q'}\cdot\boldsymbol{q}_{j'})+(\boldsymbol{q}_{q'}\cdot\boldsymbol{k})(\boldsymbol{q}_{j'}\cdot\boldsymbol{q}_{p'}) \tag{8.24}$$

Depending on what values vectors $\boldsymbol{q}_{j'},\boldsymbol{q}_{p'},\boldsymbol{q}_{q'}$ take, we can distinguish the following cases:

$$\mathcal{D}_{j'p'q'}(\boldsymbol{k}) = \begin{cases} 3, & \text{in } \textbf{case 1}, \\ 1, & \text{in } \textbf{case 2}, \\ \boldsymbol{k}_1^{\perp}\cdot\boldsymbol{k}_2^{\perp}, & \text{in } \textbf{case 3}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.25}$$

Case 1 takes place if all vectors $\boldsymbol{q}_{j'},\boldsymbol{q}_{p'},\boldsymbol{q}_{q'}$ are the same and are equal to $\boldsymbol{k}$, that is the triplet $\{\boldsymbol{q}_{j'},\boldsymbol{q}_{p'},\boldsymbol{q}_{q'}\} = \{\boldsymbol{k},\boldsymbol{k},\boldsymbol{k}\}$.

In **case 2**, one of the vectors from the triplet equals to $\boldsymbol{k}$, and the remaining two vectors are the same and are different from $\boldsymbol{k}$, that is the triplet $\{\boldsymbol{q}_{j'},\boldsymbol{q}_{p'},\boldsymbol{q}_{q'}\}$ is either equal to $\{\boldsymbol{k},\boldsymbol{k}_r^{\perp},\boldsymbol{k}_r^{\perp}\}$ or $\{\boldsymbol{k}_r^{\perp},\boldsymbol{k},\boldsymbol{k}_r^{\perp}\}$, or else to $\{\boldsymbol{k}_r^{\perp},\boldsymbol{k}_r^{\perp},\boldsymbol{k}\}$, where $r = 1, 2$.

In **case 3**, similarly as in **case 2**, one of the vectors from the triplet is equal to $\boldsymbol{k}$ but the remaining two are different from $\boldsymbol{k}$ and are also different from each other, that is the triplet $\{\boldsymbol{q}_{j'}, \boldsymbol{q}_{p'}, \boldsymbol{q}_{q'}\}$ is equal to either $\{\boldsymbol{k}, \boldsymbol{k}_1^\perp, \boldsymbol{k}_2^\perp\}$, or $\{\boldsymbol{k}, \boldsymbol{k}_2^\perp, \boldsymbol{k}_1^\perp\}$, or $\{\boldsymbol{k}_1^\perp, \boldsymbol{k}, \boldsymbol{k}_2^\perp\}$, or $\{\boldsymbol{k}_2^\perp, \boldsymbol{k}, \boldsymbol{k}_1^\perp\}$, or $\{\boldsymbol{k}_1^\perp, \boldsymbol{k}_2^\perp, \boldsymbol{k}\}$, or $\{\boldsymbol{k}_2^\perp, \boldsymbol{k}_1^\perp, \boldsymbol{k}\}$.

Remark 8.2. If we choose vectors $\boldsymbol{k}_1^\perp$, and $\boldsymbol{k}_2^\perp$ so that they are perpendicular to each other, then obviously $\mathcal{D}_{j'p'q'}(\boldsymbol{k}) = 0$ also in **case 3** *(see (8.24)).*

Assuming that $\boldsymbol{k}_1^\perp \cdot \boldsymbol{k}_2^\perp = 0$, taking into account the formulas (8.23) and (8.24) and the fact that $\lambda_L + 2\mu_L \equiv c_l^2$ is the squared longitudinal wave speed, while $\mu_L \equiv c_s^2$ is the squared shear wave speed, we finally conclude that the nonzero interacting coefficients are

$$\varGamma^j_{pq} = \begin{cases} \mp\dfrac{3}{2}c_l, & \text{if} \quad \boldsymbol{q}_{j'} = \boldsymbol{q}_{p'} = \boldsymbol{q}_{q'} = \boldsymbol{k}, \\ \mp\dfrac{1}{2}c_l, & \text{if} \quad \{\boldsymbol{q}_{j'}, \boldsymbol{q}_{p'}, \boldsymbol{q}_{q'}\} = \{\boldsymbol{k}, \boldsymbol{k}_r^\perp, \boldsymbol{k}_r^\perp\}, \\ \mp\dfrac{1}{2}\dfrac{c_l^2}{c_s}, & \text{if} \quad \{\boldsymbol{q}_{j'}, \boldsymbol{q}_{p'}, \boldsymbol{q}_{q'}\} = \{\boldsymbol{k}_r^\perp, \boldsymbol{k}, \boldsymbol{k}_r^\perp\} \quad \text{or} \quad = \{\boldsymbol{k}_r^\perp, \boldsymbol{k}_r^\perp, \boldsymbol{k}\}. \end{cases} \tag{8.26}$$

Corollary 8.3. *We have shown that in case of a geometrically nonlinear but physically linear isotropic elastic medium, we can express all wave interaction coefficients entirely in terms of the (linearized at a zero constant state) longitudinal and shear waves speeds.*

8.3.2 Interaction Coefficients for a Nonlinear Isotropic Material

Having calculated the interaction coefficients for a physically linear material, we now treat a *general isotropic material*[1] around the trivial unperturbed state $\boldsymbol{d} = 0$. As we took $\boldsymbol{\pi} = \boldsymbol{I}$, this amounts to the assumption of a weak disturbance on an undeformed unstressed material.

We carry out the calculations as above, but with general isotropic energy

$$W(\boldsymbol{E}) = \widetilde{W}(\boldsymbol{J}_{\boldsymbol{E}}) \quad \text{where} \quad \boldsymbol{J}_{\boldsymbol{E}} = \{I_{\boldsymbol{E}}, II_{\boldsymbol{E}}, III_{\boldsymbol{E}}\}.$$

As we have noted, the strain vanishes at the background state,

[1] that is both geometrically and physically nonlinear material.

$$\boldsymbol{E}|_{\boldsymbol{d}=\boldsymbol{0}} = \boldsymbol{0},$$

which will lead to many simplifications. In particular, as the coefficients involve only three derivatives of W, it suffices to treat the Murnaghan form of the energy (8.15). Moreover, the terms in (8.15) are homogeneous of order two and three with respect to $\boldsymbol{E}$, and thus if we differentiate W twice and set $\boldsymbol{d} = 0$, only the quadratic terms persist. This implies that as above, we get the same flux matrix (8.22) as for a physically linear material,

$$\boldsymbol{B}(\boldsymbol{0}, \boldsymbol{k}) = (\lambda_L + 2\,\mu_L)\,\boldsymbol{k} \otimes \boldsymbol{k} + \mu_L\,(\boldsymbol{I} - \boldsymbol{k} \otimes \boldsymbol{k}),$$

which thus has the same eigensystem as above. Furthermore, the quantities $\boldsymbol{\mathcal{P}}_{\boldsymbol{q}}$ and $\boldsymbol{\mathcal{Q}}_{jpq}$ are the same as the previous case, and the terms involving $D^2_{\boldsymbol{E}}\,W$, (connected with geometrical nonlinearity) are also the same. The only change to the coefficients is that the terms with $D^3_{\boldsymbol{E}}\,W$ (connected with physical nonlinearity), no longer vanish. In order to find $D^3_{\boldsymbol{E}}\,W$, we need to calculate $D^3_{\boldsymbol{E}}\,(I_{\boldsymbol{E}})^3$, $D^3_{\boldsymbol{E}}\,(I_{\boldsymbol{E}} I\!I_{\boldsymbol{E}})$ and $D^3_{\boldsymbol{E}}\,I\!I\!I_{\boldsymbol{E}}$. Similarly as in (8.17), we obtain

$$D^3_{\boldsymbol{E}}\,(I_{\boldsymbol{E}})^3(\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3) = 6\,I_{\boldsymbol{P_1}}\,I_{\boldsymbol{P_2}}\,I_{\boldsymbol{P_3}}, \tag{8.27}$$

$$\text{and} \quad D^3_{\boldsymbol{E}}\,(I_{\boldsymbol{E^3}})(\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3) = 6\,I_{\boldsymbol{P_1 P_2 P_3}}. \tag{8.28}$$

Please recall that here and below we use the notation

$$I_{\boldsymbol{P_j}} \equiv tr\,\boldsymbol{P}_j.$$

Next we will prove two short lemmas:

Lemma 8.4. *We have*

$$D^3_{\boldsymbol{E}}\,(I_{\boldsymbol{E}} I\!I_{\boldsymbol{E}})(\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3) = 3\,I_{\boldsymbol{P_1}}\,I_{\boldsymbol{P_2}}\,I_{\boldsymbol{P_3}} - \sum\,I_{\boldsymbol{P_i}}\,I_{\boldsymbol{P_j}\,\boldsymbol{P_k}}, \tag{8.29}$$

where the sum is over ordered permutations of $\{i, j, k\} = \{1, 2, 3\}$.

Proof:

$$D_{\boldsymbol{E}}(I_{\boldsymbol{E}} I\!I_{\boldsymbol{E}})(\boldsymbol{P}) = I_{\boldsymbol{P}} I\!I_{\boldsymbol{E}} + I_{\boldsymbol{E}}(I_{\boldsymbol{E}} I_{\boldsymbol{P}} - I_{\boldsymbol{E}\,\boldsymbol{P}}), \tag{8.30}$$

$$\begin{aligned} D^2_{\boldsymbol{E}}\,(I_{\boldsymbol{E}} I\!I_{\boldsymbol{E}})(\boldsymbol{P}_1, \boldsymbol{P}_2) = {} & I_{\boldsymbol{P_1}}(I_{\boldsymbol{E}} I_{\boldsymbol{P_2}} - I_{\boldsymbol{E}\,\boldsymbol{P_2}}) + \\ & I_{\boldsymbol{P_2}}(I_{\boldsymbol{E}} I_{\boldsymbol{P_1}} - I_{\boldsymbol{E}\,\boldsymbol{P_1}}) + \\ & I_{\boldsymbol{E}}(I_{\boldsymbol{P_2}} I_{\boldsymbol{P_1}} - I_{\boldsymbol{P_2}\,\boldsymbol{P_1}}), \end{aligned} \tag{8.31}$$

$$\begin{aligned} D^3_{\boldsymbol{E}}\,(I_{\boldsymbol{E}} II_{\boldsymbol{E}})(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) &= I_{\boldsymbol{P}_1}(I_{\boldsymbol{P}_3} I_{\boldsymbol{P}_2} - I_{\boldsymbol{P}_3\,\boldsymbol{P}_2}) + \\ &\quad I_{\boldsymbol{P}_2}(I_{\boldsymbol{P}_1} I_{\boldsymbol{P}_3} - I_{\boldsymbol{P}_1\,\boldsymbol{P}_3}) + \\ &\quad I_{\boldsymbol{P}_3}(I_{\boldsymbol{P}_2} I_{\boldsymbol{P}_1} - I_{\boldsymbol{P}_2\,\boldsymbol{P}_1}) \\ &= 3\, I_{\boldsymbol{P}_1}\, I_{\boldsymbol{P}_2}\, I_{\boldsymbol{P}_3} - \sum I_{\boldsymbol{P}_i}\, I_{\boldsymbol{P}_j\,\boldsymbol{P}_k}, \end{aligned} \tag{8.32}$$

hence we get (8.29).

Lemma 8.5. *We have*

$$\begin{aligned} D^3_{\boldsymbol{E}}\, III_{\boldsymbol{E}}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) &= 2 I_{\boldsymbol{P}_1\,\boldsymbol{P}_2\,\boldsymbol{P}_3} + I_{\boldsymbol{P}_1} I_{\boldsymbol{P}_2} I_{\boldsymbol{P}_3} \\ &\quad - \sum I_{\boldsymbol{P}_i}\, I_{\boldsymbol{P}_j\,\boldsymbol{P}_k}, \end{aligned} \tag{8.33}$$

where, as in Lemma 8.4, the sum is over ordered permutations of $\{1,2,3\}$.

Proof: From Cayley-Hamilton equation, we get

$$\boldsymbol{E}^3 - I_{\boldsymbol{E}} \boldsymbol{E}^2 + II_{\boldsymbol{E}} \boldsymbol{E} - III_{\boldsymbol{E}} \boldsymbol{I} = \boldsymbol{0}.$$

Taking the trace, we obtain

$$\begin{aligned} tr\boldsymbol{E}^3 \equiv I_{\boldsymbol{E}^3} &= I_{\boldsymbol{E}}(I_{\boldsymbol{E}^2} - II_{\boldsymbol{E}}) + 3III_{\boldsymbol{E}} \\ &= (I_{\boldsymbol{E}})^3 - 3 I_{\boldsymbol{E}} II_{\boldsymbol{E}} + 3III_{\boldsymbol{E}}. \end{aligned} \tag{8.34}$$

Therefore

$$III_{\boldsymbol{E}} = \frac{1}{3} I_{\boldsymbol{E}^3} - \frac{1}{3}(I_{\boldsymbol{E}})^3 + I_{\boldsymbol{E}} II_{\boldsymbol{E}}.$$

Taking into account (8.27), (8.28) and (8.29), we get

$$\begin{aligned} D^3_{\boldsymbol{E}}\, III_{\boldsymbol{E}}(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) &= 2(I_{\boldsymbol{P}_1\,\boldsymbol{P}_2\,\boldsymbol{P}_3} - I_{\boldsymbol{P}_1} I_{\boldsymbol{P}_2} I_{\boldsymbol{P}_3}) + \\ &\quad 3\, I_{\boldsymbol{P}_1}\, I_{\boldsymbol{P}_2}\, I_{\boldsymbol{P}_3} - \sum I_{\boldsymbol{P}_i}\, I_{\boldsymbol{P}_j\,\boldsymbol{P}_k} \end{aligned} \tag{8.35}$$

and hence (8.33).

Differentiating (8.15) three times, and collecting (8.27), (8.29), and (8.33), we get after some algebra

$$\begin{aligned} D^3_{\boldsymbol{E}}\, W(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3) &= 2\,(l_M + 2\,m_M)\, I_{\boldsymbol{P}_1}\, I_{\boldsymbol{P}_2}\, I_{\boldsymbol{P}_3} \\ &\quad + (2\,m_M - n_M) \sum I_{\boldsymbol{P}_i}\, I_{\boldsymbol{P}_j\,\boldsymbol{P}_k} \\ &\quad + 2\, n_M I_{\boldsymbol{P}_1\,\boldsymbol{P}_2\,\boldsymbol{P}_3} \end{aligned} \tag{8.36}$$

where the sum, as it was before, is taken over ordered permutations of $\{1,2,3\}$.

Similarly to the formulas (8.19) and (8.20), we can derive the following identities:

$$\begin{aligned} tr\,(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_j}\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_p}\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_q}) \quad &= \tfrac{1}{4}[(\boldsymbol{q}_j\cdot\boldsymbol{k})\,(\boldsymbol{q}_p\cdot\boldsymbol{k})\,(\boldsymbol{q}_q\cdot\boldsymbol{k}) + (\boldsymbol{q}_j\cdot\boldsymbol{q}_p)\,(\boldsymbol{q}_q\cdot\boldsymbol{k}) + \\ &\qquad (\boldsymbol{q}_p\cdot\boldsymbol{q}_q)\,(\boldsymbol{q}_j\cdot\boldsymbol{k}) + (\boldsymbol{q}_q\cdot\boldsymbol{q}_j)\,(\boldsymbol{q}_p\cdot\boldsymbol{k})], \\ tr\,\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_j} tr\,\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_p} tr\,\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_q} &= (\boldsymbol{q}_j\cdot\boldsymbol{k})\,(\boldsymbol{q}_p\cdot\boldsymbol{k})\,(\boldsymbol{q}_q\cdot\boldsymbol{k}), \\ tr\,\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_j} tr\,(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_p}\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_q}) &= \frac{1}{2}(\boldsymbol{q}_j\cdot\boldsymbol{k})\,\left[(\boldsymbol{q}_p\cdot\boldsymbol{k})\,(\boldsymbol{q}_q\cdot\boldsymbol{k}) + (\boldsymbol{q}_p\cdot\boldsymbol{q}_q)\right]. \end{aligned} \tag{8.37}$$

Now, let $\boldsymbol{P}_1 \equiv \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_j}$, $\boldsymbol{P}_2 \equiv \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_p}$, $\boldsymbol{P}_3 \equiv \boldsymbol{\mathcal{P}}_{\boldsymbol{q}_q}$. Taking into account (8.37), we have from (8.36), after some algebra

$$D^3_{\boldsymbol{E}}\,W(\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_j},\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_p},\boldsymbol{\mathcal{P}}_{\boldsymbol{q}_q}) = (2\,l_M + m_M)\,\mathcal{C}_{jpq}(\boldsymbol{k}) + m_M\,\mathcal{D}_{jpq}(\boldsymbol{k}), \tag{8.38}$$

where

$$\mathcal{C}_{jpq}(\boldsymbol{k}) \equiv (\boldsymbol{q}_j\cdot\boldsymbol{k})\,(\boldsymbol{q}_p\cdot\boldsymbol{k})\,(\boldsymbol{q}_q\cdot\boldsymbol{k}),$$

and recall from (8.24) that $\mathcal{D}_{jpq}(\boldsymbol{k})$ is defined by

$$\mathcal{D}_{jpq}(\boldsymbol{k}) \equiv (\boldsymbol{q}_j\cdot\boldsymbol{q}_p)\,(\boldsymbol{q}_q\cdot\boldsymbol{k}) + (\boldsymbol{q}_p\cdot\boldsymbol{q}_q)\,(\boldsymbol{q}_j\cdot\boldsymbol{k}) + (\boldsymbol{q}_q\cdot\boldsymbol{q}_j)\,(\boldsymbol{q}_p\cdot\boldsymbol{k}).$$

The formula (8.38) shows contribution to the value of interaction coefficients (8.8) caused by physical nonlinearities. Combining (8.38) with (8.8) and (8.23) we obtain the coefficients for a fully nonlinear isotropic material calculated at the undisturbed state $\boldsymbol{d} = \boldsymbol{0}$,

$$2\,\lambda_j\,\Gamma^j_{pq} = (2\,l_M + m_M)\,\mathcal{C}_{j'p'q'}(\boldsymbol{k}) + (\lambda_L + 2\,\mu_L + m_M)\,\mathcal{D}_{j'p'q'}(\boldsymbol{k}). \tag{8.39}$$

Recalling our eigenvectors, we note that $\mathcal{C}_{j'p'q'}(\boldsymbol{k})$ vanishes unless $\boldsymbol{q}_{j'} = \boldsymbol{q}_{p'} = \boldsymbol{q}_{q'} = \boldsymbol{k}$. Taking into account the formula (8.25) we have

$$\Gamma^j_{pq} = \begin{cases} \mp\left(\dfrac{3}{2}c_l + \dfrac{2m_M + l_M}{c_l}\right) & \text{if} \quad \boldsymbol{q}_{j'} = \boldsymbol{q}_{p'} = \boldsymbol{q}_{q'} = \boldsymbol{k}, \\ \mp\dfrac{1}{2}\left(c_l + \dfrac{m_M}{c_l}\right) & \text{if} \quad \{\boldsymbol{q}_{j'},\boldsymbol{q}_{p'},\boldsymbol{q}_{q'}\} = \{\boldsymbol{k},\boldsymbol{k}^{\perp}_r,\boldsymbol{k}^{\perp}_r\}, \\ \mp\dfrac{1}{2}\left(\dfrac{c_l^2 + m_M}{c_s}\right) & \text{if} \quad \{\boldsymbol{q}_{j'},\boldsymbol{q}_{p'}\boldsymbol{q}_{q'}\} = \{\boldsymbol{k}^{\perp}_r,\boldsymbol{k},\boldsymbol{k}^{\perp}_r\} \text{ or} \\ & \qquad \{\boldsymbol{k}^{\perp}_r,\boldsymbol{k}^{\perp}_r,\boldsymbol{k}\}. \end{cases} \tag{8.40}$$

Remark 8.6. The formula (8.40) has been obtained under the assumption that $\rho_0 \equiv 1$. Without this simplifying assumption this formula changes only slightly. For completeness, we display here formula (8.40) for the interaction coefficients with *arbitrary* $\rho_0 > 0$: [2]

$$\Gamma^j_{pq} = \begin{cases} \mp\left(\dfrac{3}{2}\sqrt{\dfrac{\lambda_L + 2\mu_L}{\rho_0}} + \dfrac{l_M + 2m_M}{\rho_0}\sqrt{\dfrac{\rho_0}{\lambda_L + 2\mu_L}}\right) & \text{in case (1)},\\ \mp\dfrac{1}{2}\left(\sqrt{\dfrac{\lambda_L + 2\mu_L}{\rho_0}} + \dfrac{m_M}{\rho_0}\sqrt{\dfrac{\rho_0}{\lambda_L + 2\mu_L}}\right) & \text{in case (2)},\\ \mp\dfrac{1}{2}\left(\dfrac{\lambda_L + 2\mu_L}{\rho_0}\sqrt{\dfrac{\rho_0}{\mu_L}} + \dfrac{m_M}{\rho_0}\sqrt{\dfrac{\rho_0}{\mu_L}}\right) & \text{in case (3)}. \end{cases}$$

Corollary 8.7. *Physical nonlinearities bring additional dependence (in comparison to the physically linear material) of the interaction coefficients on two, third order Murnaghan elastic constants l_M and m_M. Therefore all interaction coefficients for a general isotropic material are express in terms of:*

- *two second order Lame constants λ_L and μ_L,*
- *two third order Murnaghan elastic constants l_M and m_M, and*
- *the density in the reference configuration ρ_0.*

[2] we denote the three cases in formula (8.40) as case (1), case (2) and case (3).

9

Asymptotics for an Isotropic Medium

In this chapter we apply the method of weakly nonlinear asymptotics to the isotropic plane waves elastodynamics (5.19) or (9.2).

Assuming that direction of wave propagation $\boldsymbol{k} = [1, 0, 0]$ that is along the positive direction of the x - axis,[1] we derive the asymptotic evolution equations for waves' amplitudes. Two cases of initial data are considered. We investigate a Cauchy problem with compact support initial data, and separately the Cauchy problem with periodic initial data.

9.1 Quasilinear Equations for the Murnaghan Material

We assume that the energy function describing the isotropic material is given by the Murnaghan form [126] (c.f. (6.7)):

$$\begin{aligned} W = &\frac{1}{2}(\lambda_L + 2\mu_L){I_{\mathbf{E}}}^2 - 2\mu_L II_{\mathbf{E}} + \\ &\frac{1}{3}(l_M + 2m_M){I_{\mathbf{E}}}^3 - 2m_M I_{\mathbf{E}} II_{\mathbf{E}} + n_M III_{\mathbf{E}}. \end{aligned} \tag{9.1}$$

The above form of the strain energy is the most general for our analysis of the isotropic media since higher than third order terms are of no importance in the analysis. We briefly recall the results obtained in Sec. 7.2.1 presenting them in a slightly modified form. Following the procedure described in Sec. 5.5 we arrive at the quasilinear system (9.2) for plane waves:

[1] as before we relabel the 1D material coordinate as x.

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{A}(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x} = \boldsymbol{0}, \tag{9.2}$$

where

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{v}(x,t) \\ \boldsymbol{d}(x,t) \end{bmatrix}, \qquad \boldsymbol{A}(\boldsymbol{w}) = -\begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}(\boldsymbol{d}) \\ \boldsymbol{I} & \boldsymbol{0}, \end{bmatrix} \tag{9.3}$$

and $\boldsymbol{B}$ is the symmetric matrix with components

$$V_{,ij} = \frac{\partial^2 V(\boldsymbol{d})}{\partial d_i \partial d_j},$$

where $V(\boldsymbol{d})$ is related to W by the formula (5.17). We recall our assumption that the density in the reference configuration $\rho_0 = 1$. Truncating the expression for the energy function so that the stress tensor has only quadratically nonlinear terms we obtain the matrix $\boldsymbol{B}(\boldsymbol{d})$ in the following form:

$$\boldsymbol{B}(\boldsymbol{d}) = \begin{bmatrix} \alpha_1 + \beta_1 d_1 & \beta_2 d_2 & \beta_2 d_3 \\ \beta_2 d_2 & \alpha_2 + \beta_2 d_1 & 0 \\ \beta_2 d_3 & 0 & \alpha_2 + \beta_2 d_1 \end{bmatrix} \tag{9.4}$$

with

$$\begin{aligned} \alpha_1 &= \lambda_L + 2\mu_L, \\ \alpha_2 &= \mu_L, \\ \beta_1 &= 3(\lambda_L + 2\mu_L) + 2(l_M + 2m_M), \\ \beta_2 &= \lambda_L + 2\mu_L + m_M. \end{aligned}$$

The eigenvalues of this matrix $\boldsymbol{B}(\boldsymbol{d})$ are

$$\begin{aligned} \kappa_1 &= \frac{1}{2}\left[\lambda_L + 3\,\mu_L + (2\,l_M + 5\,m_M + 4\,\lambda_L + 8\,\mu_L)\,d_1 + \sqrt{\delta(\boldsymbol{d})}\right], \\ \kappa_2 &= \frac{1}{2}\left[\lambda_L + 3\,\mu_L + (2\,l_M + 5\,m_M + 4\,\lambda_L + 8\,\mu_L)\,d_1 - \sqrt{\delta(\boldsymbol{d})}\right], \\ \kappa_3 &= \mu_L + (m_M + \lambda_L + 2\,\mu_L)\,d_1, \end{aligned}$$

where

$$\delta(\boldsymbol{d}) = [\lambda_L + \mu_L + (2\, l_M + 3\, m_M + 2\, \lambda_L + 4\, \mu_L) d_1]^2 + 4\, (m_M + \lambda_L + 2\, \mu_L)^2\, (d_2^2 + d_3^2).$$

The corresponding right eigenvectors:

$$\boldsymbol{q}_1 = \left[\frac{\lambda_L + \mu_L + (2\, l_M + 3\, m_M + 2\, \lambda_L + 4\, \mu_L)\; d_1 + \sqrt{\delta(\boldsymbol{d})}}{2\, (m_M + \lambda_L + 2\, \mu_L)}, d_2, d_3 \right],$$

$$\boldsymbol{q}_2 = \left[\frac{\lambda_L + \mu_L + (2\, l_M + 3\, m_M + 2\, \lambda_L + 4\, \mu_L)\; d_1 - \sqrt{\delta(\boldsymbol{d})}}{2\, (m_M + \lambda_L + 2\, \mu_L)}, d_2, d_3 \right],$$

$$\boldsymbol{q}_3 = [\, 0, -d_3, d_2] \,.$$

Let us remind that from Lemma 5.11 the pairs $\{\lambda_j, \boldsymbol{r}_j\}$ from the eigensystem of the whole matrix $\boldsymbol{A}$ are related to the pairs $\{\kappa_i, \boldsymbol{q}_i\}$ from the eigensystem of the matrix $\boldsymbol{B}$ as follows,

$$\lambda_1 = -\sqrt{\kappa_1} = -\lambda_2,$$
$$\lambda_3 = -\sqrt{\kappa_2} = -\lambda_4,$$
$$\lambda_5 = -\sqrt{\kappa_3} = -\lambda_6,$$

$$\boldsymbol{r}_1 = \begin{bmatrix} -\sqrt{\kappa}_1 \boldsymbol{q}_1 \\ \boldsymbol{q}_1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{r}_2 = \begin{bmatrix} \sqrt{\kappa}_1 \boldsymbol{q}_1 \\ \boldsymbol{q}_1 \end{bmatrix},$$

$$\boldsymbol{r}_3 = \begin{bmatrix} -\sqrt{\kappa}_2 \boldsymbol{q}_2 \\ \boldsymbol{q}_2 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{r}_4 = \begin{bmatrix} \sqrt{\kappa}_2 \boldsymbol{q}_2 \\ \boldsymbol{q}_2 \end{bmatrix},$$

$$\boldsymbol{r}_5 = \begin{bmatrix} -\sqrt{\kappa}_3 \boldsymbol{q}_3 \\ \boldsymbol{q}_3 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{r}_6 = \begin{bmatrix} \sqrt{\kappa}_3 \boldsymbol{q}_3 \\ \boldsymbol{q}_3 \end{bmatrix}.$$

Therefore taking into account the form of the eigenvectors q_i, we conclude that the first pair $\{\kappa_1, \boldsymbol{q}_1\}$ corresponds to the pair of quasi–longitudinal waves characterized by the speeds $\pm\lambda_1$ and polarizations $\boldsymbol{r}_1, \boldsymbol{r}_2$. The second pair $\{\kappa_2, \boldsymbol{q}_2\}$ corresponds to the quasi–shear waves propagating with the speeds $\pm\lambda_3$ and polarizations $\boldsymbol{r}_3, \boldsymbol{r}_4$. Finally the third pair $\{\kappa_3, \boldsymbol{q}_3\}$ is related to the pair of pure shear waves whose speeds are $\pm\lambda_5$ and polarizations $\boldsymbol{r}_5, \boldsymbol{r}_6$.

Now we calculate the gradients of eigenvectors with respect to the vector $\boldsymbol{d}$. We have

$$\nabla\kappa_1 = \frac{1}{2}\left[2l_M + 5m_M + 4\lambda_L + 8\mu_L + \frac{(\nabla\delta(\boldsymbol{d}))}{2\sqrt{\delta(\boldsymbol{d})}}\right],$$

$$\nabla\kappa_2 = \frac{1}{2}\left[2l_M + 5m_M + 4\lambda_L + 8\mu_L - \frac{(\nabla\delta(\boldsymbol{d}))}{2\sqrt{\delta(\boldsymbol{d})}}\right],$$

$$\nabla\kappa_3 = [\, m_M + \lambda_L + 2\mu_L,\ 0, 0\,],$$

here

$$\begin{aligned}\nabla(\delta(\boldsymbol{d})) &= [\,(\delta(\boldsymbol{d}))_{,d_1},\ (\delta(\boldsymbol{d}))_{,d_2},\ (\delta(\boldsymbol{d}))_{,d_3}] \\ &= 2\,[\,\lambda_L + \mu_L + (2\,l_M + 3\,m_M + 2\,\lambda_L + 4\,\mu_L)\ d_1, \\ &\quad 4(m_M + \lambda_L + 2\,\mu_L)^2 d_2, 4(m_M + \lambda_L + 2\,\mu_L)^2 d_3].\end{aligned}$$

One can easily notice that $\nabla\kappa_3 \cdot \boldsymbol{q}_3 = 0$ for any $\boldsymbol{d}$, hence the pure shear waves propagating with the speeds λ_5 and λ_6 are *globally linearly degenerate.*

At $\boldsymbol{d} = \boldsymbol{0}$, we have

$$\nabla\kappa_1 \cdot \boldsymbol{q}_1\big|_{\boldsymbol{d}=\boldsymbol{0}} = \frac{(2l_M + 4m_M + 3\lambda_L + 6\mu_L\,)\,(\lambda_L + \mu_L)}{m_M + \lambda_L + 2\mu_L},$$

while

$$\nabla\kappa_2 \cdot \boldsymbol{q}_2\big|_{\boldsymbol{d}=\boldsymbol{0}} = 0.$$

Hence the quasi-longitudinal waves are in general *locally genuinely non-linear* while quasi–shear waves are *locally linearly degenerate.*

9.2 Matrix $\boldsymbol{A}(\boldsymbol{0})$ for an Isotropic Material.

The matrix $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{k}) = \boldsymbol{A}(\boldsymbol{0})$ is

$$\boldsymbol{A}(\boldsymbol{0}) = -\begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}(\boldsymbol{0}) \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \tag{9.5}$$

where

$$\boldsymbol{B}(\mathbf{0}) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix}, \tag{9.6}$$

with

$$\alpha_1 = \lambda_L + 2\mu_L,$$

$$\alpha_2 = \mu_L.$$

The eigenvalues of the matrix $\boldsymbol{A}(\mathbf{0})$ look as follows: [2]

$$\lambda_1 = -\sqrt{\alpha_1} = -\lambda_2,$$
$$\lambda_3 = -\sqrt{\alpha_2} = -\lambda_4,$$
$$\lambda_5 = -\sqrt{\alpha_3} = -\lambda_6.$$

Right Eigenvectors:

$$\boldsymbol{r}_1 = [-\lambda_1,\, 0,\, 0,\, 1,\, 0,\, 0], \qquad \text{(longitudinal)}$$
$$\boldsymbol{r}_2 = [-\lambda_2,\, 0,\, 0,\, 1,\, 0,\, 0], \qquad \text{(longitudinal)}$$
$$\boldsymbol{r}_3 = [0,\, -\lambda_3,\, 0,\, 0,\, 1,\, 0], \qquad \text{(transverse)}$$
$$\boldsymbol{r}_4 = [0,\, -\lambda_4,\, 0,\, 0,\, 1,\, 0], \qquad \text{(transverse)}$$
$$\boldsymbol{r}_5 = [0,\, 0,\, -\lambda_5,\, 0,\, 0,\, 1], \qquad \text{(transverse)}$$
$$\boldsymbol{r}_6 = [0,\, 0,\, -\lambda_6,\, 0,\, 0,\, 1]. \qquad \text{(transverse)}$$

Left Eigenvectors:

[2] Please recall that we have set $\varrho_0 \equiv 1$, so actually without this assumption the eigenvalues are

$$\lambda_1 = -\sqrt{\frac{\lambda_L + 2\mu_L}{\rho_0}} = -\lambda_2,$$
$$\lambda_3 = -\sqrt{\frac{\mu_L}{\rho_0}} = -\lambda_4,$$
$$\lambda_5 = -\sqrt{\frac{\mu_L}{\rho_0}} = -\lambda_6.$$

$$
\begin{aligned}
\boldsymbol{l}_1 &= \tfrac{1}{2}[-\lambda_1^{-1}, 0, 0, 1, 0, 0], && \text{(longitudinal)}\\
\boldsymbol{l}_2 &= \tfrac{1}{2}[-\lambda_2^{-1}, 0, 0, 1, 0, 0], && \text{(longitudinal)}\\
\boldsymbol{l}_3 &= \tfrac{1}{2}[0, -\lambda_3^{-1}, 0, 0, 1, 0], && \text{(transverse)}\\
\boldsymbol{l}_4 &= \tfrac{1}{2}[0, -\lambda_4^{-1}, 0, 0, 1, 0], && \text{(transverse)}\\
\boldsymbol{l}_5 &= \tfrac{1}{2}[0, 0, -\lambda_5^{-1}, 0, 0, 1], && \text{(transverse)}\\
\boldsymbol{l}_6 &= \tfrac{1}{2}[0, 0, -\lambda_6^{-1}, 0, 0, 1]. && \text{(transverse)}
\end{aligned}
$$

We assume that always $\lambda_L + 2\mu_L > 0$, and $\mu_L > 0$.

9.3 Expansion

Let us consider the Cauchy problem with perturbed initial data:

$$
\begin{cases}
\boldsymbol{L}\boldsymbol{w}^\epsilon = 0, \\
\boldsymbol{w}^\epsilon{}_{|\,t=0} = \epsilon\, \boldsymbol{w}_1(x, x/\epsilon)
\end{cases}
\tag{9.7}
$$

with

$$
\boldsymbol{L} = \partial_t + \boldsymbol{A}(\boldsymbol{w}^\epsilon)\partial_x.
$$

Here ϵ is a small parameter, and

$$
\boldsymbol{w}^\epsilon = \begin{bmatrix} \boldsymbol{v}^\epsilon(t,x) \\ \boldsymbol{d}^\epsilon(t,x) \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{w}^\epsilon) = -\begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}(\boldsymbol{d}^\epsilon) \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}
\tag{9.8}
$$

where $\boldsymbol{B}$ is defined by (9.4).

9.3.1 Nonresonant Case

In this section we derive the asymptotic evolution equations for small amplitude noninteracting weakly nonlinear elastic waves. The form of the evolution equations depends heavily on the properties of the eigenstructure of the system. Two properties, namely loss of strict hyperbolicity and loss of genuine nonlinearity which occur for shear and quasi-shear waves are crucial in defining the asymptotics. While standard geometric optics expansion suffices to derive nonlinear evolution equations for the longitudinal waves, modifications of the classical expansion are needed to obtain nonlinear evolution equations for transverse waves.

Assuming that the initial data are of compact support, we recall that the classical weakly nonlinear asymptotic expansion described in Sec. 3.2 looks as follows:

$$\boldsymbol{w}^\epsilon(t,x) = \epsilon \sum_{j=1}^{6} \sigma_j(t, x, \frac{x-\lambda_j t}{\epsilon})\, \boldsymbol{r}_j + \mathcal{O}(\epsilon^2), \tag{9.9}$$

where σ_j are the unknown amplitudes and $\{\lambda_i, \boldsymbol{r}_i\}$ is the eigensystem of $\boldsymbol{A}(\boldsymbol{0})$.

We plug the expansion (9.9) into the system from (9.7) and (9.8). Using multiple scale analysis and employing the solvability condition we obtain the transport evolution equations for the amplitudes σ_j. As it was derived in Sec. 3.2 (see (3.7)), each of these equations has the following form:

$$\sigma_{j,t} + \lambda_j \sigma_{j,x} + \mathit{\Gamma}_j \sigma_j \sigma_{j,\eta} = 0.$$

We have shown in Sec. 7.1.1 that at $\boldsymbol{d} = 0$ longitudinal waves corresponding to the amplitudes σ_1 and σ_2 are genuinely nonlinear, hence $\mathit{\Gamma}_j = \mathit{\Gamma}^j_{jj} \neq 0$ for $j = 1, 2$. On the other hand, we know also from Sec. 7.1.1 that $\mathit{\Gamma}^j_{jj} = 0$ for shear and quasi-shear waves corresponding to the amplitudes $\sigma_3, \sigma_4, \sigma_5$, and σ_6. Therefore the asymptotic nonresonant equations for the elastic wave amplitudes look as follows:

$$\begin{aligned}
&\sigma_{1,t} + \lambda_1 \sigma_{1,x} + \mathit{\Gamma}_1 \sigma_1 \sigma_{1,\eta} = 0 \\
&\sigma_{2,t} + \lambda_2 \sigma_{2,x} + \mathit{\Gamma}_2 \sigma_2 \sigma_{2,\eta} = 0 \\
&\sigma_{3,t} + \lambda_3 \sigma_{3,x} = 0 \\
&\sigma_{4,t} + \lambda_4 \sigma_{4,x} = 0 \\
&\sigma_{5,t} + \lambda_5 \sigma_{5,x} = 0 \\
&\sigma_{6,t} + \lambda_6 \sigma_{6,x} = 0.
\end{aligned} \tag{9.10}$$

Hence we see that the amplitudes of shear waves are described by linear transport equations. However on the longer time scale and with different scaling of a small parameter, the nonlinearity in the evolution equations for shear waves can be revealed. This will be done in the next section. Here we concentrate on longitudinal waves. Quadratic type of nonlinearity is dominant for longitudinal waves. The canonical transport evolution equations derived with the help of classical *WNGO* expansion for longitudinal waves, in all cases considered here, have the same form (3.7) with $j = 1, 2$, where

$$\lambda_1 = -\sqrt{\alpha_1} = -\sqrt{\lambda_L + 2\mu_L} = -\lambda_2,$$
$$\Gamma_1 = -\Gamma_2 = -\frac{\beta_1}{2\sqrt{\alpha_1}} = \frac{3(\lambda_L + 2\mu_L) + 2(l_M + 2m_M)}{2\sqrt{\lambda_L + 2\mu_L}}. \tag{9.11}$$

The eigenvalues and the self–interacting coefficients for the St. Venant–Kirchhoff material are expressed as

$$\lambda_1 = -c_l = -\lambda_2, \;\; \Gamma_1 = -\Gamma_2 = -\frac{3c_l}{2}.$$

Therefore for the St. Venant–Kirchhoff material, the system (9.10) can be written as

$$\begin{aligned}
\sigma_{1,t} - c_l\sigma_{1,x} - \frac{3c_l}{2}\sigma_1\sigma_{1,\eta} &= 0\\
\sigma_{2,t} + c_l\sigma_{2,x} + \frac{3c_l}{2}\sigma_2\sigma_{2,\eta} &= 0\\
\sigma_{3,t} - c_s\sigma_{3,x} &= 0\\
\sigma_{4,t} + c_s\sigma_{4,x} &= 0\\
\sigma_{5,t} - c_s\sigma_{5,x} &= 0\\
\sigma_{6,t} + c_s\sigma_{6,x} &= 0.
\end{aligned} \tag{9.12}$$

From (9.10) and (9.12) we conclude that for initial data of compact support we obtain decoupled asymptotic equations – separate equation for each of the elastic wave's amplitudes. There are two nonlinear evolution asymptotic equations with quadratic nonlinearity of Burgers type for the longitudinal waves, and four linear evolution asymptotic equations for shear elastic waves. This is because the longitudinal elastic waves are locally genuinely nonlinear while shear elastic waves are locally linearly degenerate. In the next section we will use the modified asymptotic expansion from Sec. 3.4 to obtain the nonlinear evolution equations for transverse elastic waves.

9.3.2 Modified Expansion for Shear Waves

Here we use the modified asymptotics from Sec. 3.4 to treat nonlinear transverse elastic waves in the isotropic medium. We recall that shear elastic waves correspond to the eigensystem of matrix $\boldsymbol{A}(\mathbf{0})$ with subscripts 3,4,5 and 6. For these waves a local loss of genuine nonlinearity or a local loss of strict hyperbolicity typically occurs at the zero constant state. Hence cubic instead of quadratic nonlinearities decide about the evolution of such waves. In order to extract these cubic nolinearities in the evolution equations we need to change a scaling of the small parameter ϵ. Now η will be equal to

$$\eta = \frac{\phi_s}{\epsilon^2} = \frac{x - \lambda_s t}{\epsilon^2}.$$

We use the modified expansion taking into account that there are double eigenvalues:

$$\boldsymbol{w}^\epsilon(t,x) = \epsilon \left(\sigma_s(t,x,\frac{\phi_s}{\epsilon^2})\, \boldsymbol{r}_s + \sigma_{s+2}(t,x,\frac{\phi_{s+2}}{\epsilon^2})\, \boldsymbol{r}_{s+2} \right) + \mathcal{O}(\epsilon^2), \quad s = 3,4.$$

We derive again *decoupled* but now modified Burgers equations

$$\frac{\partial \sigma_s}{\partial t} + \lambda_s \frac{\partial \sigma_s}{\partial x} + \frac{1}{3} G_s \frac{\partial {\sigma_s}^3}{\partial \eta} = 0 \tag{9.13}$$

where G_s is given by the general formula (3.25). These equations have cubic instead of a quadratic nonlinearity which appears in the evolution equations for longitudinal elastic waves. The explicit values of the coefficients in (9.13) are as follows: $\lambda_3 = \lambda_5 = -\sqrt{\mu_L} = -\lambda_4 = -\lambda_6$, and

$$G_s = \frac{3}{4\lambda_s} \left(-\frac{(\lambda_L + 2\mu_L + m_M)^2}{\lambda_L + \mu_L} + \lambda_L + 2\mu_L + 2m_M \right), \quad s = 3,4,5,6. \tag{9.14}$$

Hence all coefficients in the evolution equation (9.13) are entirely expressed in terms of the second order and third order elastic constants. Please note that the only third order elastic constant appearing in (9.13) is m_M.

9.4 Nonlinear Interactions

Here we study resonant quadratically nonlinear interactions of elastic waves in an isotropic medium. The most important quantities are the interaction coefficients measuring which wave interact and how strong the interactions are.

9.4.1 Tables of Interaction Coefficients

There are three (as to the absolute value) different interaction coefficients for isotropic elastodynamics. For the physically linear material, that is when

$$W = \frac{1}{2}(\lambda_L + 2\mu_L){I_{\mathbf{E}}}^2 - 2\mu_L II_{\mathbf{E}},$$

these coefficients are expressed entirely in terms of the waves speeds of the linearized elastic waves:

$$\begin{aligned} \Gamma^1_{11} &= -\frac{3\,c_l}{2}, \\ \Gamma^1_{33} &= -\frac{c_l}{2}, \\ \Gamma^3_{13} &= -\frac{{c_l}^2}{2\,c_s}. \end{aligned}$$

When the strain energy (in Murnaghan's form) contains also third order elastic constants and is given by

$$W = \frac{1}{2}(\lambda_L + 2\mu_L){I_{\mathbf{E}}}^2 - 2\mu_L II_{\mathbf{E}} + \frac{1}{3}(l_M + 2m_M){I_{\mathbf{E}}}^3 - 2m_M I_{\mathbf{E}} II_{\mathbf{E}} + n_M III_{\mathbf{E}},$$

then the nonzero interaction coefficients look as follows:

$$\begin{aligned} \Gamma^1_{11} &= -\frac{3\,c_l^2 + 2l_M + 4m_M}{2c_l}, \\ \Gamma^1_{33} &= -\frac{c_l^2 + m_M}{2c_l}, \\ \Gamma^3_{13} &= -\frac{{c_l}^2 + m_M}{2\,c_s}. \end{aligned}$$

Hence apart from the second order constants they depend also on two third order elastic constants m and n.

The other nonzero coefficients are related by the following formulas:

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^1_{22} = -\Gamma^2_{11} = -\Gamma^2_{12} = -\Gamma^2_{21} = -\Gamma^2_{22}$$

$$\Gamma^1_{33} = \Gamma^1_{34} = \Gamma^1_{43} = \Gamma^1_{44} = -\Gamma^2_{33} = -\Gamma^2_{34} = -\Gamma^2_{43} = -\Gamma^2_{44} =$$
$$\Gamma^1_{55} = \Gamma^1_{56} = \Gamma^1_{65} = \Gamma^1_{66} = -\Gamma^2_{55} = -\Gamma^2_{56} = -\Gamma^2_{65} = -\Gamma^2_{66}$$

$$\Gamma^3_{13} = \Gamma^3_{14} = \Gamma^3_{23} = \Gamma^3_{24} = -\Gamma^4_{13} = -\Gamma^4_{14} = -\Gamma^4_{23} = -\Gamma^4_{24} =$$
$$\Gamma^3_{31} = \Gamma^3_{41} = \Gamma^3_{32} = \Gamma^3_{42} = -\Gamma^4_{31} = -\Gamma^4_{41} = -\Gamma^4_{32} = -\Gamma^4_{42} =$$
$$\Gamma^5_{15} = \Gamma^5_{16} = \Gamma^5_{25} = \Gamma^5_{26} = -\Gamma^6_{15} = -\Gamma^6_{16} = -\Gamma^6_{25} = -\Gamma^6_{26} =$$
$$\Gamma^5_{51} = \Gamma^5_{61} = \Gamma^5_{52} = \Gamma^5_{62} = -\Gamma^6_{51} = -\Gamma^6_{61} = -\Gamma^6_{52} = -\Gamma^6_{62}.$$

All these coefficients are represented in Fig. 9.1.–9.3., where

$$\boxed{\text{a}} \leftrightarrow \Gamma^1_{11}, \qquad \boxed{\text{b}} \leftrightarrow \Gamma^1_{33}, \quad \boxed{\text{d}} \leftrightarrow \Gamma^3_{13}\ .$$

We have displayed only the coefficients Γ^l_{pq} for $l = 1, 3, 5$, because the coefficients Γ^{l+1}_{pq} for $l = 1, 3, 5$ are obtained by the formula

$$\Gamma^l_{pq} = -\Gamma^{l+1}_{pq} \tag{9.15}$$

valid for any $l = 1, 3, 5$, and $p, q = 1, 2, ..., 6$, which follows from the structure of elastodynamics equations.

9.4.2 Resonant Interaction Coefficients

The fundamental feature of a nonlinear resonance of waves is the generation of a new wave with a fixed wave number and frequency being the combination of the componential wave numbers and frequencies. It is of great practical importance to investigate whether and when such nonlinear resonances take place.

Let $j \neq p \neq q \neq j$. The strength of the j-th wave produced through the nonlinear resonant interaction of p-th and q-th waves is represented by the three-wave resonant interaction coefficient Γ^j_{pq}. We recall here its definition

$$\Gamma^j_{pq}(\boldsymbol{w}_0) = \boldsymbol{l}_j \cdot [(\nabla_{\boldsymbol{w}}\, \boldsymbol{A}(\boldsymbol{w})\, \boldsymbol{r}_p)\, \boldsymbol{r}_q]\Big|_{\boldsymbol{w}=\boldsymbol{w_0}}\ . \tag{9.16}$$

Hence the the three-wave resonant interaction coefficients are these for which $j \neq p \neq q \neq j$.

6					b	b
5					b	b
4			b	b		
3			b	b		
2	a	a				
1	a	a				
	1	2	3	4	5	6

Fig. 9.1. Γ^1_{pq} - interaction coefficients for the first wave.

6						
5						
4	d	d				
3	d	d				
2			d	d		
1			d	d		
	1	2	3	4	5	6

Fig. 9.2. Γ^3_{pq} - interaction coefficients for the third wave.

6	d	d				
5	d	d				
4						
3						
2					d	d
1					d	d
	1	2	3	4	5	6

Fig. 9.3. Γ^5_{pq} - interaction coefficients for the fifth wave.

All interacting coefficients for the isotropic medium.

9.4.3 Resonant Triads

Here we consider resonant interactions of three waves. Let us clarify what we mean by such interactions. Suppose there are two waves with frequencies and wave numbers (ω_1, k_1) and (ω_2, k_2) which interact resonantly and as a result they produce a third wave with the frequency and wavenumber (ω_3, k_3) being linear combinations of the previous waves. The simplest resonant conditions for three waves (satisfying the relations $\omega_j = \lambda_j k_j$ for $j = 1, 2, 3$) are the following:

$$\begin{aligned} \omega_1 + \omega_2 + \omega_3 &= 0, \\ k_1 + k_2 + k_3 &= 0. \end{aligned}$$

We display the equations for the resonant triads in two cases:

- a) interaction of two shear $(3,4)$ – waves and the longitudinal (1) – wave, and
- b) interaction of two shear $(5,6)$ – waves and the longitudinal (1) – wave.

Similarly we obtain resonant equations for interacting triads of $(2,3,4)$ – waves and $(2,5,6)$ – waves.

In case a) the resonant equations look as follows:

$$\begin{aligned} &a_{1,t} + \lambda_1 a_{1,x} + \mathit{\Gamma}^1_{11} a_1 a_{1,\eta} + \mathit{\Gamma}^1_{34}(a_3 * a_4)_{,\eta} = 0 \\ &a_{3,t} + \lambda_3 a_{3,x} + \mathit{\Gamma}^3_{14}(a_1 * a_4)_{,\eta} = 0 \\ &a_{4,t} + \lambda_4 a_{4,x} + \mathit{\Gamma}^4_{13}(a_1 * a_3)_{,\eta} = 0, \end{aligned}$$

where $(a_p * a_q)$ denotes the integral term in (3.26) of the appropriate averaged amplitudes.

Disregarding the third order elastic constants, we obtain:

$$\begin{aligned} &a_{1,t} - c_l a_{1,x} - \frac{3}{2} c_l a_1 a_{1,\eta} - \frac{c_l}{2}(a_3 * a_4)_{,\eta} = 0 \\ &a_{3,t} - c_s a_{3,x} - \frac{c_l^2}{2c_s}(a_1 * a_4)_{,\eta} = 0 \\ &a_{4,t} + c_s a_{4,x} + \frac{c_l^2}{2c_s}(a_1 * a_3)_{,\eta} = 0. \end{aligned}$$

In case b) - interaction of two shear (5, 6) – waves and the longitudinal (1) – wave, the equations for resonant triads look as follows:

$$\begin{aligned} &a_{1,t} + \lambda_1 a_{1,x} + \mathit{\Gamma}^1_{11} a_1 a_{1,\eta} + \mathit{\Gamma}^1_{56}(a_5 * a_6)_{,\eta} = 0 \\ &a_{5,t} + \lambda_5 a_{5,x} \qquad\qquad\qquad + \mathit{\Gamma}^5_{16}(a_1 * a_6)_{,\eta} = 0 \\ &a_{6,t} + \lambda_6 a_{6,x} \qquad\qquad\qquad + \mathit{\Gamma}^6_{15}(a_1 * a_5)_{,\eta} = 0. \end{aligned}$$

Hence, by disregarding the third order elastic constants, we have:

$$\begin{aligned} &a_{1,t} - c_l a_{1,x} - \frac{3}{2} c_l a_1 a_{1,\eta} - \frac{c_l}{2}(a_5 * a_6)_{,\eta} = 0 \\ &a_{5,t} - c_s a_{5,x} \qquad\qquad - \frac{c_l^2}{2c_s}(a_1 * a_6)_{,\eta} = 0 \\ &a_{6,t} + c_s a_{6,x} \qquad\qquad + \frac{c_l^2}{2c_s}(a_1 * a_5)_{,\eta} = 0. \end{aligned}$$

General analytical solutions of the three-wave resonant nonlinear integro-differential equations, as above, are not known. Numerical experiments performed with the integro-differential equations of nonlinear acoustics in [119] reveal an interesting interplay between the nonlinear convective and integral dispersive terms. As a result, a shock formation is prevented and cusp rarefaction singularities are observed (see [119]).

However, if we restrict ourselves to harmonic amplitudes, after reduction to semilinear partial differential equations, the three-wave resonant nonlinear integro-differential equations can be solved analytically with the help of the inverse scattering method (see e.g. [98]).

10

Anisotropic Media

Let us first consider a *general anisotropic* medium. Assuming that $W(\boldsymbol{E})$ is an analytical function, we expand it up to the third order terms in strains:

$$W(\boldsymbol{E}) = \frac{1}{2}\sum_{i,j,k,l=1}^{3} c_{ijkl}E_{ij}E_{kl} + \frac{1}{6}\sum_{i,j,k,l,m,n=1}^{3} c_{ijklmn}E_{ij}E_{kl}E_{mn} \tag{10.1}$$

where E_{ij} are the components of Lagrangian strain tensor $\boldsymbol{E}$ expressed in Cartesian coordinates. To simplify the notation, instead of the constants c_{ijkl} and c_{ijklmn}, we will use Brugger's [23] second order – c_{ij}, and third order – c_{ijk} constants, where the change of the indices is done according to the known rule:

$$\begin{array}{lll} 11 \to 1 & 22 \to 2 & 33 \to 3 \\ 23 \to 4 & 32 \to 4 & 13 \to 5 \\ 31 \to 5 & 12 \to 6 & 21 \to 6. \end{array} \tag{10.2}$$

Obviously $c_{ij} = c_{ji}$, and $c_{ijk} = c_{ikj} = c_{jik} = ...$ etc. The general anisotropic medium is therefore characterized by 21 second order and 56 third order constants. However, from the symmetry requirement this number may be greatly reduced for a specific crystal. We are interested in the most symmetric of anisotropic materials - a cubic crystal.

10.1 Cubic Crystal Classes

There are five cubic crystal classes denoted by 23, $m3$, 432, $m3m$ and $\overline{4}3m$. All of these cubic crystals posses trigonal symmetry with respect to the four cube diagonals. Next, all cubic crystals have diagonal symmetry about the three cubic axes. Finally, two of the cubic crystal classes have an additional tetragonal symmetry about the cubic axes. Each of the symmetry diminish the number of elastic stiffness constants. The number of constants for the most symmetric of cubic crystals is reduced to 3 – second order, and 6 – third order. We will be interested in the most symmetric of cubic crystals those belonging to the class $m3m$. This is motivated by the fact that many crystals appearing in nature belong to this class, e.g. all metals of cubic symmetry: Au (Gold), Fe (Iron), Ni (Nickel), Ag (Silver), also ionic crystals like e.g. $NaCl, KCl, LiF, AgCl$ and many more. For the most symmetric of cubic crystal classes there are nine strain invariants [155]:

$$
\begin{aligned}
I_1 &= E_{11} + E_{22} + E_{33}, \\
I_2 &= E_{11}\, E_{22} + E_{22}\, E_{33} + E_{11}\, E_{33}, \\
I_3 &= (E_{12})^2 + (E_{23})^2 + (E_{31})^2, \\
I_4 &= E_{11}\, E_{22}\, E_{33}, \\
I_5 &= E_{12}\, E_{23}\, E_{31}, \\
I_6 &= (E_{11} + E_{22})(E_{12})^2 + (E_{22} + E_{33})(E_{23})^2 + \\
&\quad (E_{33} + E_{11})(E_{31})^2, \\
I_7 &= (E_{12})^2 E_{23}^2 + E_{23}^2 E_{31}^2 + E_{31}^2 E_{12}^2, \\
I_8 &= (E_{12})^2 E_{11} E_{22} + {E_{23}}^2 E_{22} E_{33} + {E_{31}}^2 E_{33} E_{11}, \\
I_9 &= E_{11} E_{12}^2 E_{31}^2 + E_{22} E_{23}^2 E_{12}^2 + E_{23} E_{31}^2 E_{23}^2.
\end{aligned}
$$

Restricting our attention to the terms which are at most of the third order in strains we can express the strain energy with the help of six strain invariants.

$$
W = W(I_1, I_2, I_3, I_4, I_5, I_6).
$$

Let $\boldsymbol{e}_1 = [1,0,0], \boldsymbol{e}_2 = [0,1,0], \boldsymbol{e}_3 = [0,0,1]$ be an orthogonal basis of units vectors. We will express the strain invariants in terms of these vectors and the matrix $\boldsymbol{E}$. We have

$$
\begin{aligned}
I_1 &= (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2) + (\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3),\\
I_2 &= (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)(\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2)(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3) + (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3),\\
I_3 &= (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_2)^2 + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_3)^2 + (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_3)^2,\\
I_4 &= (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)(\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2)(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3),\\
I_5 &= (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_3)(\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_3),\\
I_6 &= [(\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1) + \boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2](\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_2)^2 +\\
&\quad [(\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2) + \boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3](\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_3)^2 +\\
&\quad [(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3) + \boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1](\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_3)^2.
\end{aligned}
$$

The explicit form of the strain energy functions for the most symmetric cubic crystal including third order terms was given by Birch [14]. We write this energy using Bruggers [23] notation for the elastic constants.

$$
\begin{aligned}
W = {}& \frac{1}{2}c_{11}(E_{11}^2 + E_{22}^2 + E_{33}^2) + c_{12}(E_{11}E_{22} + E_{22}E_{33} + E_{11}E_{33}) +\\
& 2c_{44}(E_{12}^2 + E_{23}^2 + E_{31}^2) + \frac{1}{6}c_{111}(E_{11}^3 + E_{22}^3 + E_{33}^3) +\\
& \frac{1}{2}c_{112}\{E_{11}^2(E_{22} + E_{33}) + E_{22}^2(E_{11} + E_{33}) + E_{33}^2(E_{11} + E_{22})\} +\\
& 2c_{144}\left[(E_{11}E_{23}^2 + E_{22}E_{13}^2 + E_{33}E_{12}^2)\right] +\\
& 2c_{166}\{(E_{11} + E_{22})E_{12}^2 + (E_{22} + E_{33})E_{23}^2 + (E_{11} + E_{33})E_{13}^2\} +\\
& c_{123}E_{11}E_{22}E_{33} + 8c_{456}E_{12}E_{23}E_{31}.
\end{aligned}
\tag{10.3}
$$

Here c_{ij}, and c_{ijk} are Brugger's second order and third order elastic constants, and E_{ij}, as before, are the components of Lagrangian strain tensor $\boldsymbol{E}$. Using the invariants we can write this energy as

$$
\begin{aligned}
W = & \frac{1}{2}c_{11}(I_1^2 - 2I_2) + c_{12}I_2 + 2c_{44}I_3 + \\
& \frac{1}{6}c_{111}(I_1^3 - 3I_1I_2 + 3I_4) + \frac{1}{2}c_{112}(I_1I_2 - 3I_4) + \\
& 2c_{144}(I_1I_3 - I_6) + c_{123}I_4 + 8c_{456}I_5 + 2c_{166}\, I_6.
\end{aligned}
\tag{10.4}
$$

The same energy written in another form looks as follows

$$
\begin{aligned}
W = & \frac{1}{2}c_{11}I_1^2 + (c_{12} - c_{11})I_2 + 2c_{44}I_3 + \\
& \frac{1}{6}c_{111}I_1^3 + \frac{1}{2}(c_{112} - c_{111})I_1I_2 + 2c_{144}I_1I_3 + \\
& \frac{1}{2}(c_{111} - 3c_{112} + 2c_{123})I_4 + 8c_{456}I_5 + 2(c_{166} - c_{144})\, I_6.
\end{aligned}
\tag{10.5}
$$

While in the formula (10.4) the terms are grouped according to the elastic constants, in the above formula (10.5) they are grouped with respect to the invariants.

10.1.1 Physically Linear Material

In this section we will derive the general equations for a cubic crystal of the most symmetric class for arbitrary direction but for a physically linear material. Therefore we assume for the time being that

$$
W = \frac{1}{2}c_{11}(I_1^2 - 2I_2) + c_{12}I_2 + 2c_{44}I_3.
$$

We have

$$
\begin{aligned}
(D_{\boldsymbol{E}}\, I_1)(\boldsymbol{P}) &= (\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1) + (\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2) + (\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3), \\
(D_{\boldsymbol{E}}\, {I_1}^2)(\boldsymbol{P}) &= 2I_1 D_{\boldsymbol{E}}\, I_1(\boldsymbol{P}), \\
({D_{\boldsymbol{E}}}^2 I_1)(\boldsymbol{P}, \boldsymbol{Q}) &= 2D_{\boldsymbol{E}}\, I_1(\boldsymbol{Q}) D_{\boldsymbol{E}}\, I_1(\boldsymbol{P}), \\
(D_{\boldsymbol{E}}\, I_2)(\boldsymbol{P}) &= (\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1)\,(\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2) + (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)\,(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2) + \\
&\quad (\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2)\,(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3) + \\
&\quad (\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1)\,(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3) + (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3), \\
(D_{\boldsymbol{E}}{}^2 I_2)(\boldsymbol{P}, \boldsymbol{Q}) &= (\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1)\,(\boldsymbol{e}_2 \cdot \boldsymbol{Q}\boldsymbol{e}_2) + (\boldsymbol{e}_1 \cdot \boldsymbol{Q}\boldsymbol{e}_1)\,(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2) + \\
&\quad (\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2)\,(\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_3) + (\boldsymbol{e}_2 \cdot \boldsymbol{Q}\boldsymbol{e}_2)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3) + \\
&\quad (\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1)\,(\boldsymbol{e}_3 \cdot \boldsymbol{Q}\boldsymbol{e}_3) + (\boldsymbol{e}_1 \cdot \boldsymbol{Q}\boldsymbol{e}_1)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3),
\end{aligned}
$$

$$\begin{aligned}(D_{\boldsymbol{E}}\, I_3)(\boldsymbol{P}) = 2[\,&(\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_2)(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_2) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_3)(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_3) + \\ &(\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_3)(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_3)],\\ (D_{\boldsymbol{E}}{}^2 I_3)(\boldsymbol{P}, \boldsymbol{Q}) = 2[\,&(\boldsymbol{e}_1 \cdot \boldsymbol{Q}\boldsymbol{e}_2)(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_2) + (\boldsymbol{e}_2 \cdot \boldsymbol{Q}\boldsymbol{e}_3)(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_3) + \\ &(\boldsymbol{e}_1 \cdot \boldsymbol{Q}\boldsymbol{e}_3)(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_3)].\end{aligned}$$

For a physically (materially) linear material, we have

$$(D_{\boldsymbol{E}}\, W)(\boldsymbol{P}) = \frac{1}{2} c_{11} D_{\boldsymbol{E}}\, ({I_1}^2 - 2I_2)(\boldsymbol{P}) + c_{12} D_{\boldsymbol{E}}\, (I_2)(\boldsymbol{P}) + 2c_{44} D_{\boldsymbol{E}}\, (I_3)(\boldsymbol{P}).$$

Since

$$\begin{aligned}D_{\boldsymbol{E}}\, ({I_1}^2 - 2I_2)(\boldsymbol{P}) = 2[&(\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)\,(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2)\,(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2) + \\ &(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3)],\end{aligned}$$

hence

$$\begin{aligned}(D_{\boldsymbol{E}}\, W)(\boldsymbol{P}) = \; & c_{11}[(\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)\,(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2)\,(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2) + \\ &\quad (\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3)] + \\ & c_{12}[(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1)\,(\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2) + (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)\,(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2) + \\ &\quad (\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_2)\,(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_2)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3) + \\ &\quad (\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_1)\,(\boldsymbol{e}_3 \cdot \boldsymbol{E}\boldsymbol{e}_3) + (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_1)\,(\boldsymbol{e}_3 \cdot \boldsymbol{P}\boldsymbol{e}_3)] + \\ & 4c_{44}[(\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_2)\,(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_2) + (\boldsymbol{e}_2 \cdot \boldsymbol{E}\boldsymbol{e}_3)\,(\boldsymbol{e}_2 \cdot \boldsymbol{P}\boldsymbol{e}_3) + \\ &\quad (\boldsymbol{e}_1 \cdot \boldsymbol{E}\boldsymbol{e}_3)\,(\boldsymbol{e}_1 \cdot \boldsymbol{P}\boldsymbol{e}_3)].\end{aligned}$$

We introduce the following notation. Let

$$\mathrm{A}(\boldsymbol{E}, \boldsymbol{P}) \equiv \sum_{j=1}^{3} (\boldsymbol{e}_j \cdot \boldsymbol{E}\boldsymbol{e}_j)(\boldsymbol{e}_j \cdot \boldsymbol{P}\boldsymbol{e}_j)$$

$$\mathrm{B}(\boldsymbol{E}, \boldsymbol{P}) \equiv \sum_{j=1}^{3} \sum_{k=1}^{3} (\boldsymbol{e}_j \cdot \boldsymbol{E}\boldsymbol{e}_j)(\boldsymbol{e}_k \cdot \boldsymbol{P}\boldsymbol{e}_k) \quad \text{for} \quad j \neq k,$$

$$\mathrm{C}(\boldsymbol{E},\boldsymbol{P}) \equiv \sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{e}_j \cdot \boldsymbol{E}\boldsymbol{e}_k)(\boldsymbol{e}_j \cdot \boldsymbol{P}\boldsymbol{e}_k) \quad \text{for} \quad j<k.$$

Then we have

$$(D_{\boldsymbol{E}}\, W)(\boldsymbol{P}) = \frac{1}{2}c_{11}\mathrm{A}(\boldsymbol{E},\boldsymbol{P}) \, + c_{12}\mathrm{B}(\boldsymbol{E},\boldsymbol{P}) \, + 4c_{44}\mathrm{C}(\boldsymbol{E},\boldsymbol{P}).$$

Similarly

$$(D_{\boldsymbol{E}}{}^2 W)(\boldsymbol{P},\boldsymbol{Q}) = \frac{1}{2}c_{11}\mathrm{A}(\boldsymbol{Q},\boldsymbol{P}) \, + c_{12}\mathrm{B}(\boldsymbol{Q},\boldsymbol{P}) \, + 4c_{44}\mathrm{C}(\boldsymbol{Q},\boldsymbol{P}).$$

Therefore we have

$$\begin{aligned}
\boldsymbol{p}\cdot\boldsymbol{B}\,\boldsymbol{q} \;&=\; D^2_{\boldsymbol{E}}\,W(\boldsymbol{\mathcal{P}}\boldsymbol{p},\boldsymbol{\mathcal{P}}\boldsymbol{q}) + (\boldsymbol{p}\cdot\boldsymbol{q})\; D_{\boldsymbol{E}}\,W(\boldsymbol{k}\otimes\boldsymbol{k})\\
&=\; c_{11}\mathrm{A}(\boldsymbol{\mathcal{P}}\boldsymbol{p},\boldsymbol{\mathcal{P}}\boldsymbol{q}) \, + c_{12}\mathrm{B}(\boldsymbol{\mathcal{P}}\boldsymbol{p},\boldsymbol{\mathcal{P}}\boldsymbol{q}) \, + 4c_{44}\mathrm{C}(\boldsymbol{\mathcal{P}}\boldsymbol{p},\boldsymbol{\mathcal{P}}\boldsymbol{q})\\
&+\; (\boldsymbol{p}\cdot\boldsymbol{q})\left[c_{11}\mathrm{A}(\boldsymbol{k}\otimes\boldsymbol{k},\boldsymbol{E}) \, + c_{12}\mathrm{B}(\boldsymbol{k}\otimes\boldsymbol{k},\boldsymbol{E}) + 4c_{44}\mathrm{C}(\boldsymbol{k}\otimes\boldsymbol{k},\boldsymbol{E})\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{A}(\boldsymbol{\mathcal{P}}\boldsymbol{p},\boldsymbol{\mathcal{P}}\boldsymbol{q}) &= \sum_{j=1}^{3}(\boldsymbol{e}_j\cdot\boldsymbol{\mathcal{P}}\boldsymbol{p}\,\boldsymbol{e}_j)(\boldsymbol{e}_j\cdot\boldsymbol{\mathcal{P}}\boldsymbol{q}\,\boldsymbol{e}_j)\\
&= \sum_{j=1}^{3}\left[k_j\,p_j + k_j^2(\boldsymbol{d}\cdot\boldsymbol{p})\right]\left[k_j\,q_j + k_j^2(\boldsymbol{d}\cdot\boldsymbol{q})\right],
\end{aligned}$$

$$\begin{aligned}
\mathrm{B}(\boldsymbol{\mathcal{P}}\boldsymbol{p},\boldsymbol{\mathcal{P}}\boldsymbol{q}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{e}_i\cdot\boldsymbol{\mathcal{P}}\boldsymbol{p}\,\boldsymbol{e}_i)(\boldsymbol{e}_j\cdot\boldsymbol{\mathcal{P}}\boldsymbol{q}\,\boldsymbol{e}_j)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\left[k_i\,p_i + k_i^2(\boldsymbol{d}\cdot\boldsymbol{p})\right]\left[k_j\,q_j + k_j^2(\boldsymbol{d}\cdot\boldsymbol{q})\right],
\end{aligned}$$

here the summation is for $i \neq j$, and

$$\begin{aligned}
\mathrm{C}(\boldsymbol{\mathcal{P}}\boldsymbol{p},\boldsymbol{\mathcal{P}}\boldsymbol{q}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{e}_i\cdot\boldsymbol{\mathcal{P}}\boldsymbol{p}\,\boldsymbol{e}_j)(\boldsymbol{e}_i\cdot\boldsymbol{\mathcal{P}}\boldsymbol{q}\,\boldsymbol{e}_j)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\left[\frac{1}{2}(k_i\,p_j + k_j\,p_i) + k_ik_j(\boldsymbol{d}\cdot\boldsymbol{p})\right]\\
&\qquad\qquad\left[\frac{1}{2}(k_i\,q_j + k_j\,q_i) + k_ik_j(\boldsymbol{d}\cdot\boldsymbol{q})\right],
\end{aligned}$$

where the sums are taken for $i < j$. On the other hand we have

$$\begin{aligned}\mathrm{A}(\boldsymbol{k}\otimes\boldsymbol{k},\boldsymbol{E}) &= \sum_{j=1}^{3}(\boldsymbol{e}_j\cdot(\boldsymbol{k}\otimes\boldsymbol{k})\,\boldsymbol{e}_j)(\boldsymbol{e}_j\cdot\boldsymbol{E}\,\boldsymbol{e}_j)\\ &= \sum_{j=1}^{3}k_j^2\left[k_j\,d_j+\frac{1}{2}k_j^2|\boldsymbol{d}|^2\right],\end{aligned}$$

$$\begin{aligned}\mathrm{B}(\boldsymbol{k}\otimes\boldsymbol{k},\boldsymbol{E}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{e}_i\cdot(\boldsymbol{k}\otimes\boldsymbol{k})\,\boldsymbol{e}_i)(\boldsymbol{e}_j\cdot\boldsymbol{E}\,\boldsymbol{e}_j)\\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}k_i^2\left[k_j\,d_j+\frac{1}{2}k_j^2|\boldsymbol{d}|^2\right],\end{aligned}$$

here the summation is for $i \neq j$, and

$$\begin{aligned}\mathrm{C}(\boldsymbol{k}\otimes\boldsymbol{k},\boldsymbol{E}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{e}_i\cdot(\boldsymbol{k}\otimes\boldsymbol{k})\,\boldsymbol{e}_j)(\boldsymbol{e}_i\cdot\boldsymbol{E}\,\boldsymbol{e}_j)\\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}(k_i\,k_j)\left[\frac{1}{2}(k_i\,d_j+k_j\,d_i+k_ik_j|\boldsymbol{d}|^2)\right],\end{aligned}$$

where the sums are taken for $i < j$.

Now we calculate $\boldsymbol{B}=\boldsymbol{B}(\boldsymbol{0},\boldsymbol{k})$:

$$\begin{aligned}\boldsymbol{p}\cdot\boldsymbol{B}(\boldsymbol{0},\boldsymbol{k})\boldsymbol{q} &= D_{\boldsymbol{E}}^2\,W(\boldsymbol{\mathcal{P}}\boldsymbol{p}(\boldsymbol{0}),\boldsymbol{\mathcal{P}}\boldsymbol{q}(\boldsymbol{0})) = c_{11}\mathrm{A}(\boldsymbol{\mathcal{P}}\boldsymbol{p}(\boldsymbol{0}),\boldsymbol{\mathcal{P}}\boldsymbol{q}(\boldsymbol{0}))\\ &\quad + c_{12}\mathrm{B}(\boldsymbol{\mathcal{P}}\boldsymbol{p}(\boldsymbol{0}),\boldsymbol{\mathcal{P}}\boldsymbol{q}(\boldsymbol{0})) + 4c_{44}\mathrm{C}(\boldsymbol{\mathcal{P}}\boldsymbol{p}(\boldsymbol{0}),\boldsymbol{\mathcal{P}}\boldsymbol{q}(\boldsymbol{0}))\\ &= c_{11}\boldsymbol{p}\cdot\begin{pmatrix}k_1^2 & 0 & 0\\ 0 & k_2^2 & 0\\ 0 & 0 & k_3^2\end{pmatrix}\boldsymbol{q} + c_{12}\boldsymbol{p}\cdot\begin{pmatrix}0 & k_1k_2 & k_1k_3\\ k_1k_2 & 0 & k_2k_3\\ k_1k_3 & k_2k_3 & 0\end{pmatrix}\boldsymbol{q}\\ &\quad + c_{44}\boldsymbol{p}\cdot\begin{pmatrix}1-k_1^2 & k_1k_2 & k_1k_3\\ k_1k_2 & 1-k_2^2 & k_2k_3\\ k_1k_3 & k_2k_3 & 1-k_3^2\end{pmatrix}\boldsymbol{q}.\end{aligned}$$

Therefore we have

$$\boldsymbol{B} = \begin{pmatrix} c_{11}k_1^2 + c_{44}(1-k_1^2) & (c_{12}+c_{44})k_1k_2 & (c_{12}+c_{44})k_1k_3 \\ (c_{12}+c_{44})k_1k_2 & c_{11}k_2^2 + c_{44}(1-k_2^2) & (c_{12}+c_{44})k_2k_3 \\ (c_{12}+c_{44})k_1k_3 & (c_{12}+c_{44})k_2k_3 & c_{11}k_3^2 + c_{44}(1-k_3^2) \end{pmatrix}. \quad (10.6)$$

Let $\boldsymbol{k} = [k_1, k_2, 0]$, so we assume now that $k_3 = 0$, then

$$\boldsymbol{B}(\boldsymbol{0}, \boldsymbol{k}) = \begin{pmatrix} c_{11}k_1^2 + c_{44}k_2^2 & (c_{12}+c_{44})k_1k_2 & 0 \\ (c_{12}+c_{44})k_1k_2 & c_{11}k_2^2 + c_{44}k_1^2 & 0 \\ 0 & 0 & c_{44} \end{pmatrix}. \quad (10.7)$$

The eigenvalues of this matrix are

$$\begin{aligned} \kappa_{1,2} &= \frac{1}{2}(c_{11} + c_{44} \pm \sqrt{(c_{11}-c_{44})^2 + [(c_{12}+c_{44})^2 - (c_{11}-c_{44})^2]\, sin^2 2\phi}), \\ \kappa_3 &= c_{44}. \end{aligned}$$

Here we have introduced the notation $cos\phi = \dfrac{k_1}{|\boldsymbol{k}|}$, and $sin\phi = \dfrac{k_2}{|\boldsymbol{k}|}$. Moreover, we recall that $|\boldsymbol{k}| = 1$, so in fact $k_1 = cos\phi$ and $k_2 = sin\phi$.

Remark 10.1. Since $1 - sin^2 2\phi = cos^2 2\phi$, so the first two eigenvalues from above can be written also as

$$\kappa_{1,2} = \frac{1}{2}(c_{11} + c_{44} \pm \sqrt{(c_{11}-c_{44})^2 cos^2 2\phi + (c_{12}+c_{44})^2 sin^2 2\phi}).$$

If $\phi = 0$ that is if $\boldsymbol{k} = [1, 0, 0]$ or $\phi = \pi/2$ ($\boldsymbol{k} = [0, 1, 0]$), then we have

$$\begin{aligned} \kappa_1 &= c_{11}, \\ \kappa_2 &= c_{44} = \kappa_3. \end{aligned}$$

On the other hand if $\phi = \pi/4$ ($\boldsymbol{k} = \frac{1}{\sqrt{2}}[1, 1, 0]$) or $\phi = 3\pi/4$ ($\boldsymbol{k} = \frac{1}{\sqrt{2}}[-1, 1, 0]$), then

$$\begin{aligned} \kappa_1 &= \frac{1}{2}(c_{11} + c_{12} + 2c_{44}), \\ \kappa_2 &= \frac{1}{2}(c_{11} - c_{12}), \\ \kappa_3 &= c_{44}. \end{aligned}$$

The eigenvalues $\kappa_1, \kappa_2, \kappa_3$ of the matrix (10.7) have the corresponding right eigenvectors as follows:

$$\begin{aligned}
\boldsymbol{q}_1 &= [\, \kappa_1 - (c_{11}k_2^2 + c_{44}k_1^2),\, (c_{12}+c_{44})k_1k_2,\, 0\,], && \text{(quasi–longitudinal)}\\
\boldsymbol{q}_2 &= [\, \kappa_2 - (c_{11}k_2^2 + c_{44}k_1^2),\, (c_{12}+c_{44})k_1k_2,\, 0\,], && \text{(quasi–shear)}\\
\boldsymbol{q}_3 &= [\, 0,\, 0,\, 1,\,]. && \text{(shear)}
\end{aligned}$$

10.2 Physically Nonlinear Material

Here we consider the cubic crystal with general energy function (10.3). Restricting ourselves to the third order terms we will give the explicit formulas for the equations. First, we have chosen three principal directions of the wave front propagation to illustrate different types of degeneracies connected with a local loss of strict hyperbolicity and genuine nonlinearity for elastic waves. These directions are $[1,0,0], [1,1,0]$ and $[1,1,1]$. Moreover we will also consider the case of an arbitrary direction in the plane$(1,1,0)$ (see Sec. 12.4).

10.3 Three Principal Directions in a Cubic Crystal

The three chosen directions are the only ones for which three pure modes exist - one longitudinal and two transverse.
The case **a)** $\boldsymbol{k}_1 = [1,0,0]$ defines a direction along the x axis that is the direction of the wave front.

Remark 10.2. The isotropic case is identical with the case **a)** of a cubic crystal.

Given the other direction $\boldsymbol{k}_j$ of a wave front propagation, we rotate the Cartesian coordinate system, so that the vector $\boldsymbol{k}_1 = [1,0,0]$ goes into the unit vector of the direction $\boldsymbol{k}_j$. We then need to transform the formula (10.3) for the energy $W = W(\boldsymbol{E})$ according to the rule:

$$W(\boldsymbol{E}) \longrightarrow \tilde{W}(\boldsymbol{Q}_{\boldsymbol{k}j}\boldsymbol{E}\boldsymbol{Q}_{\boldsymbol{k}j}^T) \tag{10.8}$$

where $\boldsymbol{Q}_{\boldsymbol{k}j}$ is the orthogonal matrix of the rotation that transforms the vector $\boldsymbol{k}_1 = [1,0,0]$ into the unit vector of the direction $\boldsymbol{k}_j$. The rotation matrices $\boldsymbol{Q}_{\boldsymbol{k}j}$ corresponding to the directions $\boldsymbol{k}_j$ are chosen as follows:

in the case **b)** $\boldsymbol{k}_2 = \frac{1}{\sqrt{2}}[1,1,0]$

$$\boldsymbol{Q}_{\boldsymbol{k}2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \tag{10.9}$$

and in the case **c)** $\boldsymbol{k}_3 = \frac{1}{\sqrt{3}}[1,1,1]$

$$\boldsymbol{Q}_{\boldsymbol{k}3} = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \end{bmatrix}. \tag{10.10}$$

The results of the calculations of the corresponding matrix $\mathbf{B}(\boldsymbol{d})$ are displayed in Sec. 11.4 **a)**, **b)** and **c)**, respectively.

10.4 Eigensystem of Matrix $\boldsymbol{A}(\mathbf{0})$

In this section we analyze the algebraic structure of matrix $\boldsymbol{A}(\boldsymbol{w})$, evaluated at the zero unstrained constant state $\boldsymbol{w}_0 = \mathbf{0}$. Unlike the isotropic case, the form of the elasticity equations in a cubic crystal depends on the chosen direction of propagation. Hence for three chosen directions we obtain three different matrices $\boldsymbol{A}(\boldsymbol{w})$. However at the zero constant state in all these three cases **a)**, **b)** and **c)** the structure of the eigensystems of the matrices $\boldsymbol{A}(\mathbf{0})$ coincides with the one for the isotropic case (see Sec. 9.2):

$$\boldsymbol{A}(\mathbf{0}) = - \begin{pmatrix} 0 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Eigenvalues:

$$\lambda_1 = -\sqrt{\alpha_1} = -\lambda_2,$$
$$\lambda_3 = -\sqrt{\alpha_2} = -\lambda_4,$$
$$\lambda_5 = -\sqrt{\alpha_3} = -\lambda_6.$$

The corresponding right eigenvectors are:

$$\boldsymbol{r}_1 = [\,-\lambda_1,\ 0,\ 0,\ 1,\ 0,\ 0\,], \qquad \text{(longitudinal)}$$
$$\boldsymbol{r}_2 = [\,-\lambda_2,\ 0,\ 0,\ 1,\ 0,\ 0\,], \qquad \text{(longitudinal)}$$
$$\boldsymbol{r}_3 = [\,0,\ -\lambda_3,\ 0,\ 0,\ 1,\ 0\,], \qquad \text{(transverse)}$$
$$\boldsymbol{r}_4 = [\,0,\ -\lambda_4,\ 0,\ 0,\ 1,\ 0\,], \qquad \text{(transverse)}$$
$$\boldsymbol{r}_5 = [\,0,\ 0,\ -\lambda_5,\ 0,\ 0,\ 1\,], \qquad \text{(transverse)}$$
$$\boldsymbol{r}_6 = [\,0,\ 0,\ -\lambda_6,\ 0,\ 0,\ 1\,]. \qquad \text{(transverse)}$$

The left eigenvectors have the form:

$$\boldsymbol{l}_1 = \tfrac{1}{2}[\,-\lambda_1^{-1},\ 0,\ 0,\ 1,\ 0,\ 0\,], \qquad \text{(longitudinal)}$$
$$\boldsymbol{l}_2 = \tfrac{1}{2}[\,-\lambda_2^{-1},\ 0,\ 0,\ 1,\ 0,\ 0\,], \qquad \text{(longitudinal)}$$
$$\boldsymbol{l}_3 = \tfrac{1}{2}[\,0,\ -\lambda_3^{-1},\ 0,\ 0,\ 1,\ 0\,], \qquad \text{(transverse)}$$
$$\boldsymbol{l}_4 = \tfrac{1}{2}[\,0,\ -\lambda_4^{-1},\ 0,\ 0,\ 1,\ 0\,], \qquad \text{(transverse)}$$
$$\boldsymbol{l}_5 = \tfrac{1}{2}[\,0,\ 0,\ -\lambda_5^{-1},\ 0,\ 0,\ 1\,], \qquad \text{(transverse)}$$
$$\boldsymbol{l}_6 = \tfrac{1}{2}[\,0,\ 0,\ -\lambda_6^{-1},\ 0,\ 0,\ 1\,]. \qquad \text{(transverse)}$$

We assume that always $\alpha_j > 0$,[1] hence in the three considered cases we have three pairs of real eigenvalues with mutually opposite signs. The system of elastodynamics, however, is *not always strictly hyperbolic* at $\boldsymbol{w}_0$ because in cases **a)** and **c)**, $\alpha_2 = \alpha_3$. Nevertheless, we always have a complete set of eigenvectors, expressed in terms of the corresponding eigenvalues. Moreover a distinction into pure shear and longitudinal waves is clear in these three cases. One pair of eigenvalues corresponds to the velocities of longitudinal waves (subscripts $1, 2$), while the remaining two pairs to the velocities of transverse waves (subsripts $3, 4$ and $5, 6$).

[1] The explicit values of α_j expressed in terms of Brugger's constants are given in Sec 11.4.

11

Asymptotics for a Cubic Crystal

11.1 Asymptotic Equations for Longitudinal Waves

The longitudinal waves correspond to the eigensystem of the matrix $\boldsymbol{A}(\boldsymbol{w_0})$ with subscripts 1 and 2, (see Sec. 10.4). In all cases considered here, *locally* at $\boldsymbol{w}_0 = \boldsymbol{0}$, these are *strictly hyperbolic* ($\lambda_1 \neq \lambda_2$,) and *genuinely nonlinear waves*:

$$(\nabla_{\boldsymbol{w}}\, \lambda_j(\boldsymbol{w}) \cdot \boldsymbol{r}_j)\big|_{\boldsymbol{w}\,=\,\boldsymbol{0}} \neq 0 \qquad (11.1)$$

for j=1,2; where $\lambda_j(\boldsymbol{w})$ is the eingenvalue of the matrix $\boldsymbol{A}(\boldsymbol{w})$.
A dominating type of nonlinearity for longitudinal waves is quadratic. The canonical transport evolution equations derived with the help of classical *WNGO* expansion for longitudinal waves, in all cases considered here, have the same form (3.7) with $j = 1, 2$, where

$$\lambda_1 = -\sqrt{\alpha_1} = -\lambda_2, \quad \Gamma_1 = -\Gamma_2 = -\frac{\beta_1}{2\sqrt{\alpha_1}}\,.$$

The particular values of constants α_1 and β_1 differ for different directions of waves propagation and are displayed in Sec 11.4.

11.2 Asymptotic Equations for Shear Waves

Here we illustrate the use of the modified asymptotics to treat three types of degeneracies in the examples of transverse waves in the simplest anisotropic elastic medium - the most symmetric of cubic crystals. The shear elastic waves considered here correspond to the eigensystem of matrix $\boldsymbol{A}(\boldsymbol{0})$ with subscripts 3,4,5 and 6 (see Sec.10.4). For these waves a local loss of genuine nonlinearity and/or strict hyperbolicity typically occurs at the zero constant state. The three particular directions of wave propagation were chosen to illustrate explicitly different types of such degeneracies. Let $\phi_s \equiv x - \lambda_s t$.

a) Direction $[1,0,0]$ with a unit vector $\boldsymbol{k}_1 = [1,0,0]$. In this case we have $\lambda_3 = \lambda_5 = -\sqrt{\alpha_2} = -\lambda_4 = -\lambda_6$, and

$$\Gamma_s = \boldsymbol{l}_s \cdot (\nabla_{\boldsymbol{w}} \boldsymbol{A}(\boldsymbol{w})\, \boldsymbol{r}_s)\boldsymbol{r}_s\Big|_{\boldsymbol{w}=\boldsymbol{0}} = 0, \qquad s = 3,4,5,6,$$

so we have a *locally nonstrictly hyperbolic* and a *locally linearly degenerate* case. We use the modified expansion taking into account that there are double eigenvalues:

$$\boldsymbol{w}^\epsilon(t,x) = \epsilon\left(\sigma_s(t,x,\frac{\phi_s}{\epsilon^2})\,\boldsymbol{r}_s + \sigma_{s+2}(t,x,\frac{\phi_{s+2}}{\epsilon^2})\,\boldsymbol{r}_{s+2}\right) + \mathcal{O}(\epsilon^2), \quad s = 3,4.$$

We derive again *decoupled* equations (9.13) for the amplitudes of shear waves in spite of the fact that we have two pairs of double eigenvalues. This decoupling is the result of the local loss of genuine nonliearity and the special symmetries in the system. In (9.13)

$$G_s = \frac{3}{4\lambda_s}\left(\frac{\beta_2^2}{\alpha_2 - \alpha_1} + \gamma_3\right), \quad s = 3,4,5,6. \tag{11.2}$$

Remark 11.1. Exactly the same structure have the evolution equations obtained for an isotropic case.

b) Direction $[1,1,0]$ with the unit vector $\boldsymbol{k}_2 = \dfrac{1}{\sqrt{2}}[1,1,0]$. We have $\lambda_3 \neq \lambda_4 \neq \lambda_5 \neq \lambda_6$, and

$$\Gamma_s = \boldsymbol{l}_s \cdot (\nabla_{\boldsymbol{w}} \boldsymbol{A}(\boldsymbol{w})\, \boldsymbol{r}_s)\boldsymbol{r}_s\Big|_{\boldsymbol{w}=\boldsymbol{0}} = 0, \qquad s = 3,4,5,6. \tag{11.3}$$

Hence transverse waves in this case are *locally strictly hyperbolic* and *locally linearly degenerate*. We use the ansatz of modified asymptotics for a single s-shear wave ($s = 3,4,5,6$):

$$\boldsymbol{w}^\epsilon(t,x) = \epsilon\sigma_s(t,x,\frac{\phi_s}{\epsilon^2})\,\boldsymbol{r}_s + \mathcal{O}(\epsilon^2).$$

Following the procedure from Sec. 3.2. we end up with the equations (9.13) as the canonical asymptotic evolution equations for transverse elastic waves in this case. The explicit values of the coefficients are as follows:

$$\lambda_3 = -\sqrt{\alpha_2} = -\lambda_4,$$

$$\lambda_5 = -\sqrt{\alpha_3} = -\lambda_6,$$

$$G_s = \frac{1}{4\lambda_s}\left(\frac{3\beta_2^2}{\alpha_2 - \alpha_1} + \gamma_4\right), \quad s = 3, 4, \tag{11.4}$$

$$G_s = \frac{1}{4\lambda_s}\left(\frac{3\beta_3^2}{\alpha_2 - \alpha_1} + \gamma_6\right), \quad s = 5, 6. \tag{11.5}$$

c) Direction $[1,1,1]$ with the unit vector $\boldsymbol{k}_3 = \frac{1}{\sqrt{3}}[1,1,1]$. Here we have $\lambda_3 = \lambda_5 = -\sqrt{\alpha_2} = -\lambda_4 = -\lambda_6$, and for $s = 3, 4$

$$\Gamma_s = \boldsymbol{l}_s \cdot (\nabla_{\boldsymbol{w}}\, \boldsymbol{A}(\boldsymbol{w})\, \boldsymbol{r}_s)\boldsymbol{r}_s\Big|_{\boldsymbol{w}=\boldsymbol{0}} \neq 0, \tag{11.6}$$

$$\Gamma_{s+2} = \boldsymbol{l}_{s+2} \cdot (\nabla_{\boldsymbol{w}}\, \boldsymbol{A}(\boldsymbol{w})\, \boldsymbol{r}_{s+2})\boldsymbol{r}_{s+2}\Big|_{\boldsymbol{w}=\boldsymbol{0}} = 0. \tag{11.7}$$

Hence in each pair of doubled shear waves, one is *locally linearly degenerate* and the other is *locally genuinely nonlinear*. Taking the asymptotic expansion of the form

$$\boldsymbol{w}^\epsilon(t,x) = \epsilon\left(\sigma_s(t,x,\frac{\phi_s}{\epsilon})\boldsymbol{r}_s + \sigma_{s+2}(t,x,\frac{\phi_{s+2}}{\epsilon})\,\boldsymbol{r}_{s+2}\right) + \mathcal{O}(\epsilon^2), \quad s = 3, 4,$$

we arrive at the following system of equations

$$\begin{cases} \dfrac{\partial \sigma_s}{\partial t} + \lambda_s \dfrac{\partial \sigma_s}{\partial x} + \dfrac{1}{2}\left(\Gamma^s_{s,s}\dfrac{\partial \sigma_s^2}{\partial \eta} + \Gamma^s_{s+2,s+2}\dfrac{\partial \sigma_{s+2}^2}{\partial \eta}\right) = 0 \\ \dfrac{\partial \sigma_{s+2}}{\partial t} + \lambda_{s+2}\dfrac{\partial \sigma_{s+2}}{\partial x} + \Gamma^{s+2}_{s,s+2}\dfrac{\partial(\sigma_s\sigma_{s+2})}{\partial \eta} = 0 \end{cases} \tag{11.8}$$

where λ_s are the eigenvalues of $\boldsymbol{A}(\boldsymbol{0})$ and $\Gamma^j_{p,q} = \boldsymbol{l}_j \cdot (\nabla_{\boldsymbol{w}}\, \boldsymbol{A}(\boldsymbol{w})\, \boldsymbol{r}_p\, \boldsymbol{r}_q)\Big|_{\boldsymbol{w}=\boldsymbol{0}}$.

In (11.8) we have

$$\Gamma^3_{3,3} \equiv \Gamma_3 = \frac{\beta_3}{2\sqrt{\alpha_2}} = -\Gamma^4_{4,4} \equiv -\Gamma_4,$$

$$\Gamma^s_{s,s} = -\Gamma^s_{s+2,s+2} = -\Gamma^{s+2}_{s,s+2} = -\Gamma^{s+2}_{s+2,s}.$$

Using the above relations, the system (11.8) may be written as

$$\begin{cases} \dfrac{\partial \sigma_s}{\partial t} + \lambda_s \dfrac{\partial \sigma_s}{\partial x} + \dfrac{1}{2}\Gamma_s \dfrac{\partial\left(\sigma_s^2 - \sigma_{s+2}^2\right)}{\partial \eta} = 0 \\ \dfrac{\partial \sigma_{s+2}}{\partial t} + \lambda_{s+2} \dfrac{\partial \sigma_{s+2}}{\partial x} - \Gamma_s \dfrac{\partial(\sigma_s \sigma_{s+2})}{\partial \eta} = 0. \end{cases} \tag{11.9}$$

This system of equations can be easily transformed to the *complex Burgers equation* (see (3.18) and (11.12)). This equation describes coupling at a quadratically nonlinear level between the plane shear waves when the direction of propagation is a three-fold symmetry axis [1, 1, 1]. It seems that it is for the first time these equations appear in the context of elastodynamics.

11.3 Simplified Asymptotic Equations

Here we recapitulate the canonical asymptotic equations which were derived explicitly in the previous section. After a change of the independent variables, these equations may be represented in the following simple forms (the classification is done according to the *local* behavior of the eigensystems of matrices $\mathbf{A}(\boldsymbol{w})$ at $\boldsymbol{w} = \mathbf{0}$):

- 1. *Strictly hyperbolic and genuinely nonlinear case* – canonical equations for the amplitudes σ_l of all longitudinal waves considered here:

$$(\sigma_l)_{,\tau} + \frac{1}{2}\Gamma_l(\sigma_l^2)_{,\theta} = 0. \tag{11.10}$$

- 2. *Strictly hyperbolic and locally linearly degenerate case* – canonical equations for the amplitudes σ_s of shear waves in case of [1, 1, 0] direction[1]:

$$(\sigma_s)_{,\tau} + \frac{1}{3}G_s(\sigma_s^3)_{,\theta} = 0. \tag{11.11}$$

[1] Shear waves for [1, 0, 0] direction in a cubic crystal, as well as in an isotropic medium, also belong here in spite of the local loss of strict hyperbolicity.

- 3. *Double waves case (one genuinely nonlinear and one locally linearly degenerate)* – canonical equations for the amplitudes of two shear waves σ_s and σ_{s+2} in case of $[1,1,1]$ direction:

$$\begin{cases} (\sigma_s)_{,\tau} + \dfrac{1}{2}\,\Gamma_s(\sigma_s^2 - \sigma_{s+2}^2)_{,\theta} = 0 \\ (\sigma_{s+2})_{,\tau} - \Gamma_s(\sigma_s\,\sigma_{s+2})_{,\theta} = 0. \end{cases} \tag{11.12}$$

11.4 Explicit Formulas

In this section we display the explicit forms of matrices $\mathbf{B}(\boldsymbol{d},\boldsymbol{k})$ calculated for three canonical directions of the wave front propagating in a cubic crystal. Specification of these matrices entirely determines the systems of plane waves elastodynamics in these three cases. While deriving the explicit form of each of matrices $\mathbf{B}(\boldsymbol{d},\boldsymbol{k})$, we have retained at most quadratic terms for the components of the displacement gradients. The inclusion of the quadratic terms is more than typically assumed when usually linear terms suffice. Here, however, higher order terms were needed for the derivation of the nonlinear asymptotic equations for shear waves propagating along the chosen directions (see Sec. 11.2). We have introduced the following convention in the notation of the constants appearing in matrices $\mathbf{B}(\boldsymbol{d},\boldsymbol{k})$: α_k stands at the zero order strain terms, β_k stands at the first order strain terms, and γ_k stands at the second order strain terms. We recall that

$$\mathbf{B}(\boldsymbol{d},\boldsymbol{k}) = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix}$$

where for a particular direction of wave propagation we have
a) direction with a unit vector $\boldsymbol{k} = \boldsymbol{k}_1 = [1,0,0]$:

$$B_{11} = \alpha_1 + \beta_1\ d_1 + \frac{1}{2}(\gamma_1\ {d_1}^2 + \gamma_2\ {d_2}^2 + \gamma_2\ {d_3}^2)$$
$$B_{12} = \beta_2\ d_2 + \gamma_2\ d_1\ d_2$$
$$B_{13} = \beta_2\ d_3 + \gamma_2\ d_1\ d_3$$
$$B_{22} = \alpha_2 + \beta_2\ d_1 + \frac{1}{2}(\gamma_2\ {d_1}^2 + 3\gamma_3\ {d_2}^2 + \gamma_3\ {d_3}^2)$$
$$B_{23} = \gamma_3\ d_2\ d_3$$
$$B_{33} = \alpha_2 + \beta_2\ d_1 + \frac{1}{2}(\gamma_2\ d_1^2 + \gamma_3\ d_2^2 + 3\gamma_3\ d_3^2)$$

with

$$\alpha_1 = c_{11}$$
$$\alpha_2 = c_{44}$$
$$\beta_1 = 3c_{11} + c_{111}$$
$$\beta_2 = c_{11} + c_{166}$$
$$\gamma_1 = 3(c_{11} + 2c_{111})$$
$$\gamma_2 = c_{11} + c_{111} + c_{166}$$
$$\gamma_3 = c_{11} + 2c_{166},$$

b) direction with a unit vector $\boldsymbol{k} = \boldsymbol{k}_2 = \frac{1}{\sqrt{2}}[1, 1, 0]$:

$$B_{11} = \alpha_1 + \beta_1\ d_1 + \frac{1}{2}(\gamma_1\ d_1^2 + \gamma_2\ d_2^2 + \gamma_3\ d_3^2)$$
$$B_{12} = \beta_2\ d_2 + \gamma_2\ d_1\ d_2$$
$$B_{13} = \beta_3\ d_3 + \gamma_3\ d_1\ d_3$$
$$B_{22} = \alpha_2 + \beta_2\ d_1 + \frac{1}{2}(\gamma_2\ d_1^2 + \gamma_4\ d_2^2 + \gamma_5\ d_3^2)$$
$$B_{23} = \gamma_5\ d_2\ d_3$$
$$B_{33} = \alpha_3 + \beta_3\ d_1 + \frac{1}{2}(\gamma_3\ d_1^2 + \gamma_5\ d_2^2 + \gamma_6\ d_3^2)$$

with

$$\alpha_1 = \frac{1}{2}\ (c_{11} + c_{12} + 2\ c_{44})$$
$$\alpha_2 = \frac{1}{2}\ (c_{11} - c_{12})$$
$$\alpha_3 = c_{44}$$
$$\beta_1 = \frac{3}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{1}{4}\ (c_{111} + 3c_{112} + 12\ c_{166})$$
$$\beta_2 = \frac{1}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{1}{4}\ (c_{111} - c_{112})$$
$$\beta_3 = \frac{1}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{1}{2}\ (c_{144} + c_{166} + 2\ c_{456})$$
$$\gamma_1 = \frac{3}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{3}{2}\ (c_{111} + 3c_{112} + 12\ c_{166})$$
$$\gamma_2 = \frac{1}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{1}{2}\ (c_{111} + c_{112} + 6\ c_{166})$$
$$\gamma_3 = \frac{1}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{1}{4}\ (c_{111} + 3c_{112}) + \frac{1}{4}\ (c_{144} + 7\ c_{166} + 2\ c_{456})$$
$$\gamma_4 = \frac{3}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{3}{2}\ (c_{111} - c_{112})$$
$$\gamma_5 = \frac{1}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + \frac{1}{4}\ (c_{111} - c_{112}) + \frac{1}{2}\ (c_{144} + c_{166} + 2\ c_{456})$$
$$\gamma_6 = \frac{3}{2}\ (c_{11} + c_{12} + 2\ c_{44}) + 3\ (c_{144} + c_{166} + 2c_{456}),$$

c) direction with a unit vector $\boldsymbol{k} = \boldsymbol{k}_3 = \frac{1}{\sqrt{3}}[1, 1, 1]$:

$$B_{11} = \alpha_1 + \beta_1\ d_1 + \frac{1}{2}\gamma_1\ d_1^2 + \frac{1}{2}\gamma_2\ (d_2^2 + d_3^2)$$
$$B_{12} = \beta_2\ d_2 + \gamma_2\ d_1\ d_2$$
$$B_{13} = \beta_2\ d_3 + \gamma_2\ d_1\ d_3$$
$$B_{22} = \alpha_2 + \beta_2\ d_1 - \beta_3\ d_2 + \frac{1}{2}\gamma_2\ d_1^2 + \frac{1}{2}\gamma_3\ (3\ d_2^2 + d_3^2)$$
$$B_{23} = \beta_3\ d_3 + \gamma_3\ d_2\ d_3$$
$$B_{33} = \alpha_2 + \beta_2\ d_1 + \beta_3\ d_2 + \frac{1}{2}{\gamma_4}^2 + \frac{1}{2}\gamma_3\ (d_2^2 + 3\ d_3^2)$$

with

$$
\begin{aligned}
\alpha_1 &= \frac{1}{3}(c_{11} + 2c_{12} + 4c_{44}) \\
\alpha_2 &= \frac{1}{3}(c_{11} - c_{12} + c_{44}) \\
\beta_1 &= c_{11} + 2c_{12} + 4c_{44} + \frac{1}{9}(c_{111} + 2c_{123} + 16c_{456}) + \frac{2}{3}(c_{112} + 2c_{144} + 4c_{166}) \\
\beta_2 &= \frac{1}{3}(c_{11} + 2c_{12} + 4c_{44}) + \frac{1}{9}(c_{111} - c_{123} - 2c_{456}) + \frac{1}{3}(2c_{166} - c_{144}) \\
\beta_3 &= \frac{\sqrt{2}}{18}(c_{111} + 2c_{123} - 2c_{456}) + \frac{\sqrt{2}}{6}(-c_{112} + c_{144} - c_{166}) \\
\gamma_1 &= c_{11} + 2c_{12} + 4c_{44} + \frac{2}{3}(c_{111} + 2c_{123} + 16c_{456}) + 4\ (c_{112} + 2c_{144} + 4c_{166}) \\
\gamma_2 &= \frac{1}{3}(c_{11} + 2c_{12} + 4c_{44}) + \frac{1}{9}(2c_{111} + c_{123} + 14c_{456}) + \frac{1}{3}(2c_{112} + 3c_{144} + 10c_{166}) \\
\gamma_3 &= \frac{1}{3}(c_{11} + 2c_{12} + 4c_{44}) + \frac{2}{9}(c_{111} - c_{123} - 2c_{456}) + \frac{2}{3}(2c_{166} - c_{144}).
\end{aligned}
$$

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Nonlinear Resonances in a Cubic Crystal

Let us recall that the wave interaction coefficients for the system (5.20) evaluated at a constant state $\boldsymbol{w} = \boldsymbol{0}$ are defined as follows

$$\Gamma^j_{pq} = \boldsymbol{l}_j \cdot D_{\boldsymbol{w}}\, \boldsymbol{A}(\boldsymbol{w}, \boldsymbol{k})\boldsymbol{r}_p\boldsymbol{r}_q\Big|_{\boldsymbol{w}=\boldsymbol{0}}. \tag{12.1}$$

where $\boldsymbol{l}_j$ and $\boldsymbol{r}_j$ are left and right eigenvectors of $\boldsymbol{A}(\boldsymbol{0}, \boldsymbol{k})$ normalized in such a way that $\boldsymbol{l_i} \cdot \boldsymbol{r_j} = \delta_{ij}$. These coefficients describe the strength in which p and q wave interact to produce j wave. The assumption of hyperelasticity implies that

$$\Gamma^j_{pq} = \Gamma^j_{qp} \tag{12.2}$$

for any $j, p, q = 1, 2, ..., 6$. As we have mentioned already before (see (9.15)), the structure of elastodynamics equations implies that

$$\Gamma^l_{pq} = -\Gamma^{l+1}_{pq} \tag{12.3}$$

for any $l = 1, 3, 5$, and $p, q = 1, 2, ..., 6$.

First, we have computed the explicit analytical form of *all* interaction coefficients for the system of nonlinear elastodynamic equations of a cubic crystal, at the zero constant state ($\boldsymbol{w}_0 = \boldsymbol{0}$) for three different directions of wave propagation: $[1, 0, 0]$, $[1, 1, 0]$, and $[1, 1, 1]$. In each case, the analytical form of the matrix $\boldsymbol{B}$ (see (5.22) - now with only first order terms), the eigenvalues of the matrix $\boldsymbol{A}(\boldsymbol{0})$, and the analytical formulas for the interaction coefficients are obtained. All the formulas are presented as functions of the elastic material constants c_{ij}, c_{ijk}. Moreover, in figures below a graphical representation of the interaction coefficients is included. This representation can be interpreted as follows. Each letter on the table Γ^j_{pq} denotes the nonzero value of the interaction coefficient which describes the strength of the interaction of the p wave (row) with a q wave (column) to produce j wave. For each new j-th wave, $j = 1, ..., 6$, we show a 6×6 table of interacting coefficients. Each table contain not only three-wave resonant interaction coefficients Γ^j_{pq} with $j \neq p \neq q \neq j$ but also self-interaction coefficients Γ^j_{jj} and other interaction coefficients Γ^j_{pq} with two coinciding indices, as well. The letters on each table correspond to the analytical formulas for the respective interacting coefficients. They in turn, are referred to the graphics by the letters in squares in the formulas. Because of the property (12.3) we display only the tables for Γ^j_{pq} with $j = 1, 3, 5$. Different letters correspond to different values of interacting coefficients. The empty box means that there is no interaction and the interaction coefficient is zero.

Our graphical representation helps to determine *all* the nonvanishing interacting coefficients and to find the relations between them (e.g. which of them are equal) without presenting lengthy equations. Such a visualization allows us to compare easily nonlinear resonant interaction of waves for different directions of propagation.

Let us recall (see Sec. 10.4) that the structure of the eigensystem of our matrix $\boldsymbol{A}(\boldsymbol{0})$ is preserved, irrespectively to the three chosen directions of the wave propagation $\boldsymbol{k_j}$ and is the same as in the isotropic case. There are always three pairs of eigenvalues of opposite sign:

$$\lambda_1 = -\lambda_2, \quad \lambda_3 = -\lambda_4, \quad \lambda_5 = -\lambda_6, \tag{12.4}$$

with the degeneracy $\lambda_3 = \lambda_5$ in the cases **a)** and **c)**. In each case the calculated eigenvectors of $\boldsymbol{A}(\boldsymbol{0})$ can be expressed in terms of the eigenvalues (12.4) and they take the following form:

$$\begin{aligned}
\boldsymbol{r}_1 &= [\,\lambda_1,\, 0,\, 0,\, 1,\, 0,\, 0\,], & \boldsymbol{l}_1 &= \tfrac{1}{2}[\,\lambda_1^{-1},\, 0,\, 0,\, 1,\, 0,\, 0\,],\\
\boldsymbol{r}_2 &= [\,\lambda_2,\, 0,\, 0,\, 1,\, 0,\, 0\,], & \boldsymbol{l}_2 &= \tfrac{1}{2}[\,\lambda_2^{-1},\, 0,\, 0,\, 1,\, 0,\, 0\,],\\
\boldsymbol{r}_3 &= [\,0,\, \lambda_3,\, 0,\, 0,\, 1,\, 0\,], & \boldsymbol{l}_3 &= \tfrac{1}{2}[\,0,\, \lambda_3^{-1},\, 0,\, 0,\, 1,\, 0\,],\\
\boldsymbol{r}_4 &= [\,0,\, \lambda_4,\, 0,\, 0,\, 1,\, 0\,], & \boldsymbol{l}_4 &= \tfrac{1}{2}[\,0,\, \lambda_4^{-1},\, 0,\, 0,\, 1,\, 0\,],\\
\boldsymbol{r}_5 &= [\,0,\, 0,\, \lambda_5,\, 0,\, 0,\, 1\,], & \boldsymbol{l}_5 &= \tfrac{1}{2}[\,0,\, 0,\, \lambda_5^{-1},\, 0,\, 0,\, 1\,],\\
\boldsymbol{r}_6 &= [\,0,\, 0,\, \lambda_6,\, 0,\, 0,\, 1\,], & \boldsymbol{l}_6 &= \tfrac{1}{2}[\,0,\, 0,\, \lambda_6^{-1},\, 0,\, 0,\, 1\,].
\end{aligned} \tag{12.5}$$

Formula (12.5) shows that in all the three cases analyzed we have a complete set of linearly independent eigenvectors satisfying $\boldsymbol{l_i} \cdot \boldsymbol{r_j} = \delta_{ij}$, in spite of the fact that our system is not strictly hyperbolic at $\boldsymbol{w_0} = \boldsymbol{0}$ in the cases a) and c). The completeness of the eigenvectors allows us to calculate the interaction coefficients even in these degenerate cases.

In Sec.12.4 we have included also the calculation related to the case of an arbitrary direction on the plane $(1,1,0)$ – the cube face. We denoted such direction by $[cos\phi, sin\phi, 0]$, where ϕ is an angle between the vectors $\boldsymbol{k_\phi} = [cos\phi, sin\phi, 0]$ and $\boldsymbol{k}_1 = [1,0,0]$. We have calculated explicitly the analytical form of the matrix $\boldsymbol{B}$ (see (5.22)) with first order terms, the eigenvalues of the matrix $\boldsymbol{B}(\mathbf{0})$, Since the formulas for the interaction coefficients obtained in this case are quite cumbersome, hence we present only the table of interacting coefficients which displays the structure of the relations between all nonzero coefficients.

Remark 12.1. Below we resign from the simplified assumption that $\rho_0 = 1$. Hence we recall that the relation between matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is as follows

$$\boldsymbol{A}(\boldsymbol{w},\boldsymbol{k}) = -\begin{pmatrix} \mathbf{0} & \frac{1}{\rho_0}\,\boldsymbol{B}(\boldsymbol{m},\boldsymbol{k}) \\ \boldsymbol{I} & \mathbf{0} \end{pmatrix} \tag{12.6}$$

12.1 Case a): direction $[1,0,0]$.

For the $[1,0,0]$ direction of wave propagation, the matrix $\boldsymbol{B}$ looks as follows: we have

$$\boldsymbol{B} = \begin{bmatrix} \alpha_1 + \beta_1 d_1 & \beta_2 d_2 & \beta_2 d_3 \\ \beta_2 d_2 & \alpha_2 + \beta_2 d_1 & 0 \\ \beta_2 d_3 & 0 & \alpha_2 + \beta_2 d_1 \end{bmatrix}$$

This matrix $\boldsymbol{B}$ has the same form as in the isotropic case. The constants are:

$$\alpha_1 = c_{11},$$

$$\alpha_2 = c_{44},$$

$$\beta_1 = 3c_{11} + c_{111},$$

$$\beta_2 = c_{11} + c_{166}.$$

Eigenvalues of the matrix $\boldsymbol{A}(\mathbf{0})$ look as follows:

$$\lambda_1 = -\sqrt{\frac{\alpha_1}{\rho_0}} = -\lambda_2,$$
$$\lambda_3 = -\sqrt{\frac{\alpha_2}{\rho_0}} = -\lambda_4,$$
$$\lambda_5 = -\sqrt{\frac{\alpha_2}{\rho_0}} = -\lambda_6.$$

The nonzero interaction coefficients are

$$\boxed{\text{a}} \leftrightarrow \Gamma^1_{11} = \frac{-\beta_1}{2\sqrt{\rho_0\,\alpha_1}}\ ,$$
$$\boxed{\text{b}} \leftrightarrow \Gamma^1_{33} = \frac{-\beta_2}{2\sqrt{\rho_0\,\alpha_1}}\ ,$$
$$\boxed{\text{d}} \leftrightarrow \Gamma^3_{13} = \frac{-\beta_2}{2\sqrt{\rho_0\,\alpha_2}}\ .$$

We have (see Fig. 12.1.–12.3.)

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{22} = -\Gamma^2_{11} = -\Gamma^2_{12} = -\Gamma^2_{22},$$
$$\Gamma^1_{33} = \Gamma^1_{34} = \Gamma^1_{44} = \Gamma^1_{55} = \Gamma^1_{56} = \Gamma^1_{66}$$
$$= -\Gamma^2_{33} = -\Gamma^2_{34} = -\Gamma^2_{44} = -\Gamma^2_{55} = -\Gamma^2_{56} = -\Gamma^2_{66},$$
$$\Gamma^3_{13} = \Gamma^3_{23} = \Gamma^3_{14} = \Gamma^3_{24} = \Gamma^5_{15} = \Gamma^5_{25} = \Gamma^5_{16} = \Gamma^5_{26}$$
$$= -\Gamma^4_{13} = -\Gamma^4_{23} = -\Gamma^4_{14} = -\Gamma^4_{24} = -\Gamma^6_{15} = -\Gamma^6_{25} = -\Gamma^6_{16} = -\Gamma^6_{26}.$$

12.2 Case b): direction $[1, 1, 0]$.

For the $[1, 1, 0]$ direction of wave propagation, the matrix $\boldsymbol{B}$ has the form:

$$\boldsymbol{B} = \begin{bmatrix} \alpha_1 + \beta_1 d_1 & \beta_2 d_2 & \alpha_3 d_3 \\ \beta_2 d_2 & \alpha_2 + \beta_2 d_1 & 0 \\ \alpha_3 d_3 & 0 & \alpha_3 + \beta_3 d_1 \end{bmatrix}$$

with

$$\alpha_1 = \tfrac{1}{2}(c_{11} + c_{12} + 2c_{44}),$$

$$\alpha_2 = \tfrac{1}{2}(c_{11} - c_{12}),$$

$$\alpha_3 = c_{44},$$

$$\beta_1 = \tfrac{3}{2}(c_{11} + c_{12} + 2c_{44}) + \tfrac{1}{4}(c_{111} + 3c_{112} + 12c_{166}),$$

$$\beta_2 = \tfrac{1}{2}(c_{11} + c_{12} + 2c_{44}) + \tfrac{1}{4}(c_{111} - c_{112}),$$

$$\beta_3 = \tfrac{1}{2}(c_{11} + c_{12} + 2c_{44} + c_{144} + c_{166} + 2c_{456}).$$

Eigenvalues of the matrix $\boldsymbol{A}(\mathbf{0})$:

$$\lambda_1 = -\sqrt{\frac{\alpha_1}{\rho_0}} = -\lambda_2,$$

$$\lambda_3 = -\sqrt{\frac{\alpha_2}{\rho_0}} = -\lambda_4,$$

$$\lambda_5 = -\sqrt{\frac{\alpha_3}{\rho_0}} = -\lambda_6$$

The nonzero interaction coefficients are

$$\boxed{\text{a}} \leftrightarrow \mathit{\Gamma}^1_{11} = \frac{-\beta_1}{2\sqrt{\rho_0\,\alpha_1}}\ ,$$

$$\boxed{\text{b}} \leftrightarrow \mathit{\Gamma}^1_{33} = \frac{-\beta_2}{2\sqrt{\rho_0\,\alpha_1}}\ ,$$

$$\boxed{\text{c}} \leftrightarrow \mathit{\Gamma}^1_{55} = \frac{-\beta_3}{2\sqrt{\rho_0\,\alpha_1}}\ ,$$

$$\boxed{\text{d}} \leftrightarrow \mathit{\Gamma}^3_{13} = \frac{-\beta_2}{2\sqrt{\rho_0\,\alpha_2}}\ ,$$

$$\boxed{\text{f}} \leftrightarrow \mathit{\Gamma}^5_{15} = \frac{-\beta_3}{2\sqrt{\rho_0\,\alpha_3}}\ .$$

We have (see Fig. 12.4.–12.6.)

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{22} = -\Gamma^2_{11} = -\Gamma^2_{12} = -\Gamma^2_{22},$$

$$\Gamma^1_{33} = \Gamma^1_{34} = \Gamma^1_{44} = -\Gamma^2_{33} = -\Gamma^2_{34} = -\Gamma^2_{44},$$

$$\Gamma^1_{55} = \Gamma^1_{56} = \Gamma^1_{66} = -\Gamma^2_{55} = -\Gamma^2_{56} = -\Gamma^2_{66},$$

$$\Gamma^3_{13} = \Gamma^3_{23} = \Gamma^3_{14} = \Gamma^3_{24} = -\Gamma^4_{13} = -\Gamma^4_{23} = -\Gamma^4_{14} = -\Gamma^4_{24},$$

$$\Gamma^5_{15} = \Gamma^5_{25} = \Gamma^5_{16} = \Gamma^5_{26} = -\Gamma^6_{15} = -\Gamma^6_{25} = -\Gamma^6_{16} = -\Gamma^6_{26}.$$

12.3 Case c): direction $[1, 1, 1]$.

For the $[1, 1, 1]$ direction of wave propagation, the matrix $\boldsymbol{B}$ takes the following form:

$$\boldsymbol{B} = \begin{bmatrix} \alpha_1 + \beta_1 d_1 & \beta_2 d_2 & \beta_2 d_3 \\ \beta_2 d_2 & \alpha_2 + \beta_2 d_1 - \beta_3 d_2 & \beta_3 d_3 \\ \beta_2 d_3 & \beta_3 d_3 & \alpha_2 + \beta_2 d_1 + \beta_3 d_2 \end{bmatrix}$$

with

$$\alpha_1 = \tfrac{1}{3}(c_{11} + 2c_{12} + 4c_{44}),$$

$$\alpha_2 = \tfrac{1}{3}(c_{11} - c_{12} + c_{44}),$$

$$\beta_1 = c_{11} + 2c_{12} + 4c_{44} + \tfrac{1}{9}(c_{111} + 6c_{112} + 12c_{144} + 24c_{166} + 2c_{123} + 16c_{456}),$$

$$\beta_2 = \tfrac{1}{3}(c_{11} + 2c_{12} + 4c_{44}) + \tfrac{1}{9}(c_{111} - 3c_{144} + 6c_{166} - c_{123} - 2c_{456}),$$

$$\beta_3 = \tfrac{\sqrt{2}}{18}(c_{111} - 3c_{112} + 3c_{144} - 3c_{166} + 2c_{123} - 2c_{456}).$$

The eigenvalues are:

$$\lambda_1 = -\sqrt{\frac{\alpha_1}{\rho_0}} = -\lambda_2,$$

$$\lambda_3 = -\sqrt{\frac{\alpha_2}{\rho_0}} = -\lambda_4,$$

$$\lambda_5 = -\sqrt{\frac{\alpha_2}{\rho_0}} = -\lambda_6.$$

The nonzero interaction coefficients are as follows:

$$\boxed{a} \leftrightarrow \Gamma^1_{11} = \frac{-\beta_1}{2\sqrt{\rho_0\,\alpha_1}} ,$$

$$\boxed{b} \leftrightarrow \Gamma^1_{33} = \frac{-\beta_2}{2\sqrt{\rho_0\,\alpha_1}} ,$$

$$\boxed{d} \leftrightarrow \Gamma^3_{13} = \frac{-\beta_2}{2\sqrt{\rho_0\,\alpha_2}} ,$$

$$\boxed{e} \leftrightarrow \Gamma^3_{33} = \frac{-\beta_3}{2\sqrt{\rho_0\,\alpha_2}} .$$

Moreover we assume that

$$\boxed{\bar{e}} \equiv -\boxed{e}.$$

We have (see Fig. 12.7.–12.9.)

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{22} = -\Gamma^2_{11} = -\Gamma^2_{12} = -\Gamma^2_{22},$$

$$\Gamma^1_{33} = \Gamma^1_{34} = \Gamma^1_{44} = \Gamma^1_{55} = \Gamma^1_{56} = \Gamma^1_{66}$$

$$= -\Gamma^2_{33} = -\Gamma^2_{34} = -\Gamma^2_{44} = -\Gamma^2_{55} = -\Gamma^2_{56} = -\Gamma^2_{66},$$

$$\Gamma^3_{13} = \Gamma^3_{23} = \Gamma^3_{14} = \Gamma^3_{24} = \Gamma^5_{15} = \Gamma^5_{25} = \Gamma^5_{16} = \Gamma^5_{26}$$

$$= -\Gamma^4_{13} = -\Gamma^4_{23} = -\Gamma^4_{14} = -\Gamma^4_{24} = -\Gamma^6_{15} = -\Gamma^6_{25} = -\Gamma^6_{16} = -\Gamma^6_{26},$$

$$\Gamma^3_{33} = \Gamma^3_{34} = \Gamma^3_{44} = \Gamma^3_{55} = \Gamma^3_{56} = \Gamma^3_{66} = \Gamma^5_{35} = \Gamma^5_{36} = \Gamma^5_{45} = \Gamma^5_{46}$$

$$= -\Gamma^4_{33} = -\Gamma^4_{34} = -\Gamma^4_{44} = -\Gamma^4_{55} = -\Gamma^4_{56} = -\Gamma^4_{66} = -\Gamma^6_{35} = -\Gamma^6_{36}$$

$$= -\Gamma^6_{45} = -\Gamma^6_{46}.$$

12.4 Case d): direction $[cos\phi, sin\phi, 0]$.

For the direction $\boldsymbol{k_\phi} = [cos\phi, sin\phi, 0]$ of wave propagation we have the matrix $\boldsymbol{B}$ in (5.22):

$$\boldsymbol{B}(\boldsymbol{d}) = \begin{bmatrix} \alpha_1 + \beta_1 d_1 + \beta_2 d_2 & \alpha_2 + \beta_2 d_1 + \beta_3 d_2 & \beta_4 d_3 \\ \alpha_2 + \beta_2 d_1 + \beta_3 d_2 & \alpha_3 + \beta_3 d_1 + \beta_5 d_2 & \beta_6 d_3 \\ \beta_4 d_3 & \beta_6 d_3 & \alpha_4 + \beta_7 d_1 + \beta_6 d_2 \end{bmatrix}$$

with

$$
\begin{aligned}
\alpha_1 &= \frac{1}{4}\left[(3c_{11} + c_{12} + 2c_{44}) + (c_{11} - c_{12} - 2c_{44})(cos4\phi - sin4\phi)\right],\\
\alpha_2 &= -\frac{1}{4}\left(c_{11} - c_{12} - 2c_{44}\right) sin4\phi,\\
\alpha_3 &= \frac{1}{4}\left[(c_{11} - c_{12} + 2c_{44}) - (c_{11} - c_{12} - 2c_{44})cos4\phi\right],\\
\alpha_4 &= c_{44},\\
\beta_1 &= \frac{3}{4}\left[(3c_{11} + c_{12} + 2c_{44}) + (c_{11} - c_{12} - 2c_{44})cos4\phi\right] +\\
&\quad \frac{1}{8}\left[(5c_{111} + 3c_{112} + 12c_{166}) + 3(c_{111} - c_{112} - 4c_{166})cos4\phi\right],\\
\beta_2 &= -\frac{1}{4}\left[(c_{11} - c_{12} - 2c_{44}) + (c_{111} - c_{112} - 4c_{166})\right] sin4\phi,\\
\beta_3 &= \frac{1}{4}\left[(3c_{11} + c_{12} + 2c_{44}) + (c_{11} - c_{12} - 2c_{44})cos4\phi\right] +\\
&\quad \frac{1}{8}\left[(c_{111} - c_{112} + 4c_{166}) - (c_{111} - c_{112} - 4c_{166})cos4\phi\right],\\
\beta_4 &= \frac{1}{4}\left[(3c_{11} + c_{12} + 2c_{44}) + (c_{11} - c_{12} - 2c_{44})cos4\phi\right] +\\
&\quad \frac{1}{4}\left[(c_{144} + 3c_{166} + 2c_{456}) - (c_{144} - c_{166} + 2c_{456})cos4\phi\right],\\
\beta_5 &= \frac{3}{4}\left(c_{11} + c_{12} + 2c_{44}\right) sin4\phi,\\
\beta_6 &= \frac{1}{4}\left[(-c_{11} - c_{12} - 2c_{44}) + (c_{144} - c_{166} + 2c_{456})\right] sin4\phi,\\
\beta_7 &= \frac{1}{4}\left[(3c_{11} + c_{12} + 2c_{44}) + (c_{11} - c_{12} - 2c_{44})cos4\phi\right] +\\
&\quad \frac{1}{4}\left[(c_{144} + 3c_{166} + 2c_{456}) - (c_{144} - c_{166} + 2c_{456})cos4\phi\right].
\end{aligned}
$$

Since

$$
\boldsymbol{B}(\boldsymbol{0}) = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_2 & \alpha_3 & 0 \\ 0 & 0 & \alpha_4 \end{bmatrix}
$$

hence the eigenvalues of the matrix $\boldsymbol{B}(\mathbf{0})$ look as follows:

$$\kappa_1 = \frac{1}{2}\left[\alpha_1 + \alpha_3 - \sqrt{(\alpha_1 - \alpha_3)^2 + 4\alpha_2^2}\right],$$
$$\kappa_2 = \frac{1}{2}\left[\alpha_1 + \alpha_3 - \sqrt{(\alpha_1 - \alpha_3)^2 + 4\alpha_2^2}\right],$$
$$\kappa_3 = \alpha_4.$$

We have

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{22} = -\Gamma^2_{11} = -\Gamma^2_{12} = -\Gamma^2_{22},$$
$$\Gamma^1_{33} = \Gamma^1_{34} = \Gamma^1_{44} = -\Gamma^2_{33} = -\Gamma^2_{34} = -\Gamma^2_{44},$$
$$\Gamma^1_{13} = \Gamma^1_{23} = \Gamma^1_{14} = \Gamma^1_{24} = -\Gamma^2_{13} = -\Gamma^2_{23} = -\Gamma^2_{14} = \Gamma^2_{24},$$
$$\Gamma^1_{55} = \Gamma^1_{56} = \Gamma^1_{66} = -\Gamma^2_{55} = -\Gamma^2_{56} = -\Gamma^2_{66},$$
$$\Gamma^3_{11} = \Gamma^3_{12} = \Gamma^3_{22} = -\Gamma^4_{11} = -\Gamma^4_{12} = -\Gamma^4_{22},$$
$$\Gamma^3_{13} = \Gamma^3_{23} = \Gamma^3_{14} = \Gamma^3_{24} = -\Gamma^4_{13} = -\Gamma^4_{23} = -\Gamma^4_{14} = -\Gamma^4_{24},$$
$$\Gamma^3_{33} = \Gamma^3_{34} = \Gamma^3_{44} = -\Gamma^4_{33} = -\Gamma^4_{34} = -\Gamma^4_{44},$$
$$\Gamma^3_{55} = \Gamma^3_{56} = \Gamma^3_{66} = -\Gamma^4_{55} = -\Gamma^4_{56} = -\Gamma^4_{66},$$
$$\Gamma^5_{15} = \Gamma^5_{25} = \Gamma^5_{26} = -\Gamma^6_{15} = -\Gamma^6_{25} = -\Gamma^6_{26},$$
$$\Gamma^5_{35} = \Gamma^5_{36} = \Gamma^5_{45} = \Gamma^5_{46} = -\Gamma^6_{35} = -\Gamma^6_{36} = -\Gamma^6_{45} = -\Gamma^6_{46},$$
$$\Gamma^5_{16} = -\Gamma^6_{16}.$$

12.5 Conclusions Related to Interaction Coefficients

We have calculated explicitly in a general *analytical* form all wave interaction coefficients for selected directions in a cubic crystal. We have shown that these coefficients are expressed in terms of material constants which determine the crystal.

By looking at the tables of interacting coefficients, one can notice a lot of symmetry which is characteristic for wave interactions in a cubic crystal. The number of different interaction coefficients changes from 3 for

$[1,0,0]$ direction (also in the isotropic case) to 11 for an arbitrary direction in a cube face.

The analysis of the derived formulas shows that the most important factors influencing the plane wave interactions are: the direction of wave propagation and the character of the nonlinearity of a crystal.

Beside properties (12.2) and (12.3), closer examination of the Figures leads to the following observations:

- In the cases a) and b) and c) the longitudinal waves numbered 1 and 2 can only be produced by the interaction of waves in the same pair, namely: (1,2), (3,4) or (5,6).

- In the cases a) and b) only the magnitudes of the nonlinear interactions differ.

- In the case c) (propagation along the cubic diagonal) more nonzero interaction coefficients than in the cases a) and b) occur. All the additional interaction coefficients are expressed in terms of the third order material constants. Therefore, the *physical nonlinearity* is crucial here.

- Nonlinear interactions manifest themselves already in the models with *geometrical nonlinearity* only. However, higher order, *physical* nonlinearities influence substantially the *magnitude* of the interactions.

- The interesting and unusual feature in the case d) is the fact that $\Gamma^5_{16} \neq \Gamma^5_{26}$ and $\Gamma^5_{16} \neq \Gamma^6_{16}$.

- Analytical formulas for the interaction coefficients can be useful for determining elastic constants of a crystal in suitably designed measurements.

6					b	b
5					b	b
4			b	b		
3			b	b		
2	a	a				
1	a	a				
	1	2	3	4	5	6

Fig. 12.1. Γ^1_{pq} - interaction coefficients for the first wave.

6						
5						
4	d	d				
3	d	d				
2			d	d		
1			d	d		
	1	2	3	4	5	6

Fig. 12.2. Γ^3_{pq} - interaction coefficients for the third wave.

6	d	d				
5	d	d				
4						
3						
2					d	d
1					d	d
	1	2	3	4	5	6

Fig. 12.3. Γ^5_{pq} - interaction coefficients for the fifth wave.

All interacting coefficients for the case a).

6					c	c
5					c	c
4			b	b		
3			b	b		
2	a	a				
1	a	a				
	1	2	3	4	5	6

Fig. 12.4. Γ^1_{pq} - interaction coefficients for the first wave.

6						
5						
4	d	d				
3	d	d				
2			d	d		
1			d	d		
	1	2	3	4	5	6

Fig. 12.5. Γ^3_{pq} - interaction coefficients for the third wave.

6	f	f				
5	f	f				
4						
3						
2					f	f
1					f	f
	1	2	3	4	5	6

Fig. 12.6. Γ^5_{pq} - interaction coefficients for the fifth wave.

All interacting coefficients for the case b).

6					b	b
5					b	b
4			b	b		
3			b	b		
2	a	a				
1	a	a				
	1	2	3	4	5	6

Fig. 12.7. Γ^1_{pq} - interaction coefficients for the first wave.

6					ē	ē
5					ē	ē
4	d	d	e	e		
3	d	d	e	e		
2			d	d		
1			d	d		
	1	2	3	4	5	6

Fig. 12.8. Γ^3_{pq} - interaction coefficients for the third wave.

6	d	d	e	e		
5	d	d	e	e		
4					e	e
3					e	e
2					d	d
1					d	d
	1	2	3	4	5	6

Fig. 12.9. Γ^5_{pq} - interaction coefficients for the fifth wave.

All interacting coefficients for the case c).

6					d	d
5					d	d
4	b	b	c	c		
3	b	b	c	c		
2	a	a	b	b		
1	a	a	b	b		
□	1	2	3	4	5	6

Fig. 12.10. Γ^1_{pq} - interaction coefficients for the first wave.

6					h	h
5					h	h
4	f	f	g	g		
3	f	f	g	g		
2	e	e	f	f		
1	e	e	f	f		
□	1	2	3	4	5	6

Fig. 12.11. Γ^3_{pq} - interaction coefficients for the third wave.

6	p	n	m	m		
5	n	n	m	m		
4					m	m
3					m	m
2					n	n
1					n	p
□	1	2	3	4	5	6

Fig. 12.12. Γ^5_{pq} - interaction coefficients for the fifth wave.

All interacting coefficients for the case d).

13

Conclusions and Final Remarks

13.1 Resume

In this work we have analyzed different aspects of propagation and interaction of elastic plane waves in nonlinear materials. In our analysis we have used the asymptotic method of weakly nonlinear geometric optics[1] which was appropriately modified to handle a loss of strict hyperbolicity and a loss of genuine nonlinearity occurring for (quasi)-transverse elastic waves. New results include in particular derivation of general formulas determining plane waves elastodynamics equations and describing all interaction coefficients. General considerations have been illustrated by the explicit calculations for the isotropic and cubic crystals materials.

In explicit examples for the isotropic and the most symmetric of anisotropic media, we have demonstrated the differences in the asymptotic evolution equations between longitudinal and transverse plane elastic waves. We have chosen three principal directions of wave front propagation in a cubic crystal for which, in the unstrained constant state, splitting into pure longitudinal and pure shear waves takes place. These examples illustrate different types of degeneracies occurring for shear waves. The characteristic feature of these waves is a local loss of genuine nonlinearity which is often accompanied by a local loss of strict hyperbolicity at the zero constant state. The derived evolution equations for (quasi)shear waves confirm the fact that a cubic type nonlinearity is associated with the propagations of transverse elastic waves while quadratic nonlinearity is typical for longitudinal waves. However, we have found that interest-

[1] In the context of elasticity it would be more appropriate to use rather the name: the method of weakly nonlinear geometric *acoustics*, but the name weakly nonlinear geometric *optics* is more commonly used in the literature.

ing phenomena occur for [1,1,1] direction which is an acoustic axis with a three-fold symmetry. We have proved that along this direction in each pair of doubled shear waves one is locally genuinely nonlinear and the other is locally linearly degenerate. In this case, we have derived coupled evolution equations (at a quadratically nonlinear level) for pairs of transverse waves. These equations are called *complex Burgers* equations. As it has already been emphasized, it seems that this is the first time that complex Burgers equations appear in the context of nonlinear elastodynamics.

13.2 Applications and Future Work

It is worthwhile to emphasize that our general results obtained here have a *universal character*. Once the elastic strain energy function is given, one can follow the formulated recipes, writing the equations for plane waves and finding all interaction coefficients in an *arbitrary anisotropic material*. The ideas developed in this work can be applied in the investigation of new nonlinear elastic materials like e.g. *soft tissues* or *rubber-like materials*. We have already applied some of the ideas developed in this work in different contexts like e.g. for nonlinear Maxwell system [49], nonlinear magnetoelastic waves [45] and also hyperbolic heat conduction models [46].

The growing recent interest in the research on nonlinear elastic shear waves, in particular experiments on coupling between nonlinear shear waves in a cubic crystal [66], confirms the importance of our theoretical investigations in this field. Our results on quadratically nonlinear interactions of elastic shear waves in a cubic crystal are particularly interesting in the light of recent developments on nonlinear shear waves in soft solids (see [26],[173]).

A generalization of the result presented in this work will appear in [65], where canonical evolution equations are derived describing the coupling of nonlinear quasi-shear elastic waves propagating along the acoustic axes in arbitrary anisotropic materials.

Our new formulation of the null condition, without the use of strain invariants, allows to apply it in the investigations of the properties of anisotropic materials.

In the entire work we have assumed that the initial background state is stress or strain free. However, we have done some (unpublished) calculations with prestrain data. It turns out that the direction of a prestrain is crucial. While the longitudinal prestrain introduces only quantitative changes in the structure of nonlinear wave interactions, the shear prestrain introduces substantial qualitative changes in the picture of wave interactions, completely changing in particular, the tables of interaction coefficients.

We have shown that in the analysis of nonlinear wave motion, an important role is played by the interaction coefficients. We have proved that these coefficients are expressed in terms of second order and third order elastic constants. Our formulas for the interaction coefficients can be helpful in determining the values of higher order elastic constants in a suitable designed nonlinear ultrasonic experiments. As it was demonstrated in recent experiments on the nonlinear acoustic properties of soft tissue - like solids [27] and also in the theoretical investigations [77], [173], even the fourth order elastic constants may play an important role, in particular e.g. in shear motion of soft tissues.

As we have mentioned in the introduction, one of the promising applications of the theory of nonlinear elastic wave interactions is in detection of damaged materials. The behavior of damaged materials is very sensitive to nonlinear acoustic diagnostics. Nonlinear Elastic Wave Spectroscopy (*NEWS*) method [164] is the example of a new method of nondestructive evaluation techniques. The method uses the results of nonlinear resonant wave interactions to discern damage in materials. We hope that our theoretical methods developed in this work will also be useful in this field.

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